Note Title 10/7/2013

basic

$x_1 = 2x_2$ $\quad$ x_2 free variable. $\quad$ (x_1, x_2, x_3)

$x_3 = -17x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Parametric vector equation.

A system of linear equations is <u>homogeneous</u> if it can be written as $A\vec{x} = \vec{0}$. $\quad$ A is $m \times n$ matrix.

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{x} \in \mathbb{R}^n$

$\vec{0} \in \mathbb{R}^m$

Always have trivial solution $\vec{x} = \vec{0}$ $\quad$ $\vec{0} \in \mathbb{R}^n$.

When does a non-trivial solution exist?

If every column is a pivot column

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccc|c} 1 & & & & & & 0 \\ & 1 & & & & & 0 \\ & & 1 & & & & 0 \\ & & & 1 & & & 0 \\ & & & & 1 & & 0 \\ & & & & & 1 & 0 \end{array}\right]$$

no free variables $\Rightarrow$ only trivial solution.

A has free variables $\iff$ a non-trivial solution to $A\vec{x} = \vec{0}$ exists.

$$\begin{bmatrix} 0 & 1 & 5 & 0 \\ 1 & 4 & 3 & 0 \\ 2 & 7 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -17 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable

$$x_1 - 17x_3 = 0 \qquad x_1 = 17x_3$$
$$x_2 + 5x_3 = 0. \qquad x_2 = -5x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} 17x_3 \\ -5x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 17 \\ -5 \\ 1 \end{bmatrix}$$

Set of solutions $\vec{x}$ any multiple of $\begin{bmatrix} 17 \\ -5 \\ 1 \end{bmatrix} = \vec{v}_1$

$$\text{Span}\{\vec{v}_1\}.$$

Will describe the solutions to a homogeneous system of linear equations in form: $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$.

p # vectors = # free variables.

$3x_1 - 4x_2 + 5x_3 = 0.$ $\qquad A = \begin{bmatrix} 3 & -4 & 5 & 0 \end{bmatrix}$

$\begin{array}{cccc} & \uparrow & \uparrow & \\ & \text{fr.} & \text{fr.} & \end{array}$

$x_1 = \frac{4}{3}x_2 - \frac{5}{3}x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} 4/3\,x_2 - 5/3\,x_3 \\ x_2 \\ x_3 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{fr.} \\ \leftarrow \text{fr.} \end{array}$$

$$= x_2 \overset{\vec{v}_1}{\begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \overset{\vec{v}_2}{\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}} \qquad \text{Span}\{\vec{v}_1, \vec{v}_2\}$$

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

$\begin{array}{ccccc} & \uparrow & & \uparrow & \\ & \text{fr.} & & \text{fr.} & \end{array}$

$x_1 = \underline{-2x_2 - 4x_4}$

$x_3 = -\underline{3}x_4$

$$\begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Non-homogeneous Systems.

(1)

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 2 & 0 & 4 & -1 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix} \\ \end{array}$$

$$\begin{array}{ccccc} & \uparrow & & \uparrow & 4 \\ & \text{free} & & \text{free} & \vec{b} \end{array}$$

(2)

$$x_1 = -2x_2 - 4x_4 - 1$$

$$x_3 = -3x_4 + 5$$

(3)

$$\begin{bmatrix} -2x_2 - 4x_4 - 1 \\ x_2 \\ -3x_4 + 5 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{\vec{v}_2} + \underbrace{\begin{bmatrix} -1 \\ 0 \\ 5 \\ 0 \end{bmatrix}}_{\vec{p}}$$

(4)

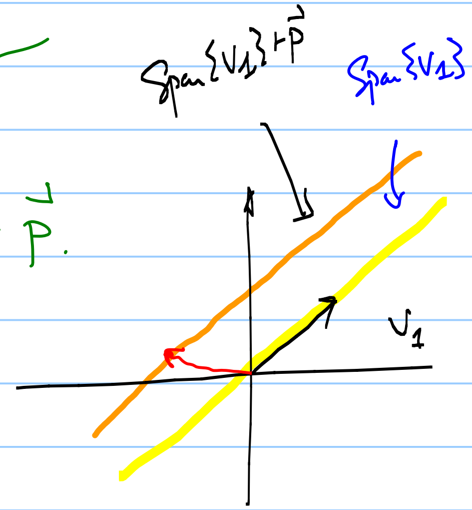
Set of all solutions: $c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{p}$

c_1, c_2 any real #'s

$$\text{Span}\{\vec{v}_1, \vec{v}_2\} + \vec{p}.$$

$\vec{v}_1 \in \mathbb{R}^2$

$$\text{Span}\{\vec{v}_1\} + \vec{p}.$$



$\text{Span}\{\vec{v}_1, \vec{v}_2\} + \vec{p} \quad \in \mathbb{R}^3$

Theorem: Suppose $A\vec{x} = \vec{b}$ is consistent.
Let $\vec{p}$ be any solution.

Then the set of all solutions $\vec{p} + V_h$

V_h: set of solutions to $A\vec{x} = \vec{0}$

Write a Solution Set in Parametric Vector form:

1. Put augmented matrix in red row echelon form.
2. Express each basic variable in terms of the free variables
3. Write a typical solution: values depend only on free variables.
4. Decompose $\vec{x}$ into a linear comb of vectors w/ free variables as weights.

$A\vec{x} = \vec{0}$ $\quad$ A $m \times n$ matrix w/ columns $\vec{a}_1 \dots \vec{a}_n$

Same as solns to vector eqn: $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$.

Definition: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent if the only solution to $x_1\vec{v}_1 + \dots + x_p\vec{v}_p = \vec{0}$ is the trivial solution.

The set is linearly dependent if there is $c_1, \dots, c_n$ (not all 0)

$$c_1\vec{v}_1 + \dots + c_p\vec{v}_p = 0.$$

Given: $\vec{v}_1 \dots \vec{v}_p$ $\quad$ $A = [\vec{v}_1 \dots \vec{v}_p] \xrightarrow{\text{Gaussian elimination}}$ are there free variables.