Note Title 10/30/2013

A $m \times n$ matrix

$\text{null}(A)$ is subspace of $\mathbb{R}^n$

$\text{col}(A)$ is subspace of $\mathbb{R}^m$.

$A \quad [\vec{a}_1 \cdots \vec{a}_n]$. $\quad \text{Col}(A) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

$U = [\vec{u}_1 \ldots \vec{u}_n]$

Suppose U is red row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\vec{u}_1 \; \vec{u}_3 \; \vec{u}_4 \; \vec{u}_6$ are lin indep.

$\text{Span}\{\vec{u}_1, \vec{u}_3, \vec{u}_4, \vec{u}_6\} = \text{Col}(A)$

Pivot cols form a basis for $\text{col}(U)$.

$\vec{u}_2 = 2\vec{u}_1$

$\vec{u}_5 = -2\vec{u}_1 + \vec{u}_3 - 3\vec{u}_4$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & -3 & 1 \\ 3 & 6 & 7 & 0 & 1 & 3 \\ -1 & -2 & 0 & 1 & -1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = U$$

for any $\vec{x} \in \mathbb{R}^6$ $\quad A\vec{x} = \vec{0}$ iff $U\vec{x} = \vec{0}$.

$\vec{a}_2 = 2\vec{a}_1$ $\quad \Longleftarrow \quad$ $\vec{u}_2 = 2\vec{u}_1$

$\vec{a}_5 = -2\vec{a}_1 + \vec{a}_3 - 3\vec{a}_4$ $\quad$ $\vec{u}_5 = -2\vec{u}_1 + \vec{u}_3 - 3\vec{u}_4$

$\text{Span}\{\vec{a}_1, \vec{a}_3, \vec{a}_4, \vec{a}_6\} = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_6\}$

Since $\vec{u}_1\ \vec{u}_3\ \vec{u}_4\ \vec{u}_6$ are lin indep

so are $\vec{a}_1\ \vec{a}_3\ \vec{a}_4\ \vec{a}_6$

basis for col (A).

row ech form indicates which cols are pivot cols.

Start w/ A $\longrightarrow$ row reduce form to find pivot cols of A.

Select pivot cols of A for the basis of col (A)

4.4 Coordinate Systems.

Unique Representation Theorem

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for vector space V. For every $\vec{x} \in V$ there is a unique set scalars $c_1, \ldots, c_n$

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

$(c_1, c_2, \ldots, c_n)$ is unique label for $\vec{x}$.

$\in \mathbb{R}^n$

Def $\mathcal{B} = \{\vec{b}_1 \cdots \vec{b}_n\}$ basis for vector space V.

Coordinates of $\vec{x}$ relative to $\mathcal{B}$ ($\mathcal{B}$- coordinates of $\vec{x}$) are $c_1 \ldots, c_n$

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n.$$

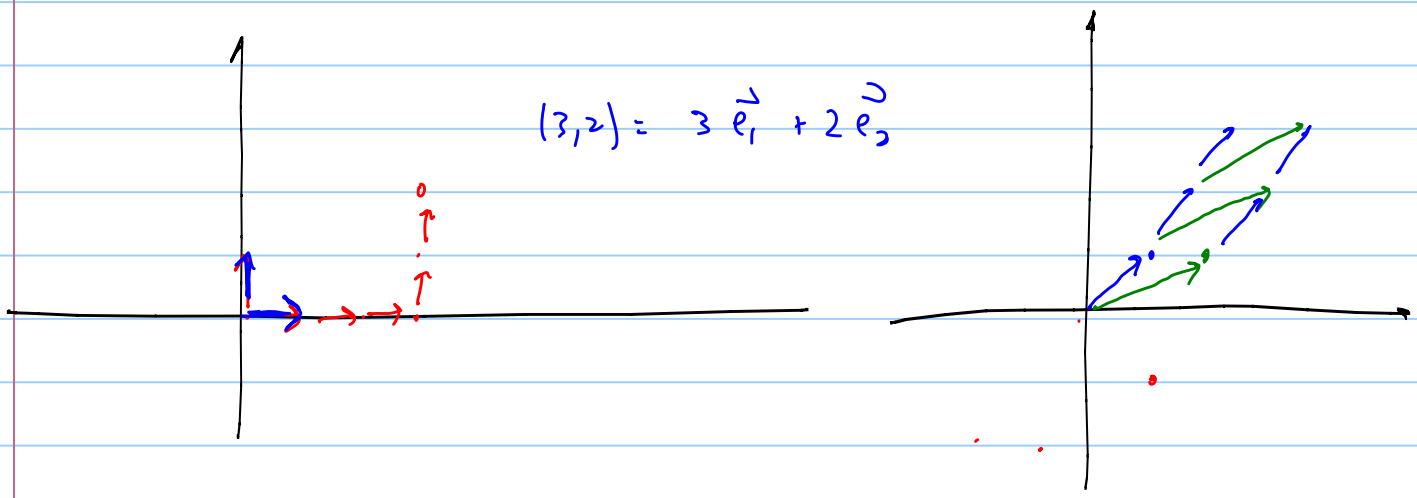
$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad \vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$$

Basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $\mathbb{R}^n$ $\qquad \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

vector $\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$-3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Example: polynomials of degree ≤ 3. $\equiv P_3$

Basis for $P_3 = \{1, t, t^2, t^3\}$

$p(t) = 3t^3 + 2t^2 - 5t + 7$ $\qquad [p]_B = \begin{bmatrix} 7 \\ -5 \\ 2 \\ 3 \end{bmatrix}$

$\mathbb{R}^2$ $\qquad B = \{\vec{b}_1 \ \vec{b}_2\}$ $\qquad \vec{b}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\qquad \vec{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\qquad [\vec{x}]_B$

Find c_1 & c_2 $\qquad c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{x}$.

$$\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$: finding — Variables

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$: given. given.

Coordinates for $\mathbb{R}^n$.

Fixed basis $\mathcal{B}$ for $\mathbb{R}^n$.

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$$

Find $\mathcal{B}$-coordinates for vector $\vec{x}$: Solve for $c_1 \cdots c_n$

$$\underbrace{\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}}_{M_{\mathcal{B}}}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$[M_{\mathcal{B}}]\,[\vec{x}]_{\mathcal{B}} = \vec{x}.$$

$[\vec{x}]_{\mathcal{B}}$

Cols of $M_{\mathcal{B}}$ are lin indep $\Rightarrow$ $M_{\mathcal{B}}$ is invertible.

$$[M_{\mathcal{B}}]^{-1}\,[M_{\mathcal{B}}]\,[\vec{x}]_{\mathcal{B}} = [M_{\mathcal{B}}]^{-1}\,\vec{x}$$

$$[\vec{x}]_{\mathcal{B}} = [M_{\mathcal{B}}]^{-1}\,\vec{x}.$$

If $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_n\}$ is a basis for V,

$\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$ is 1-1 and onto.

$\mathbb{P}_3$ polys of degree ≤ 2.

$\mathcal{B} = \{1, t, t^2\}$.

$p_1(t) = 1 + 2t^2 \qquad p_2(t) = 4 + t + 5t^2 \qquad p_3(t) = 3 + 2t$.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \vec{0}.$$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$5p_1(t) - 2p_2(t) + p_3(t) = 0 \qquad c_3 = 1 \qquad c_1 = 5$

$c_2 = -2$.