Note Title 11/25/2013

Orthogonal basis of Subspace W of $\mathbb{R}^n$. dim p

$\hookrightarrow \{\vec{u}_1, \dots, \vec{u}_p\} = \mathcal{B}$ $\qquad \vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$.

$\vec{v} \in W \qquad \vec{v} = c_1 \vec{u}_1 + \cdots + c_p \vec{u}_p$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \qquad c_i = \underbrace{\left[\frac{\vec{v}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}\right]}$$

orthogonal basis only.

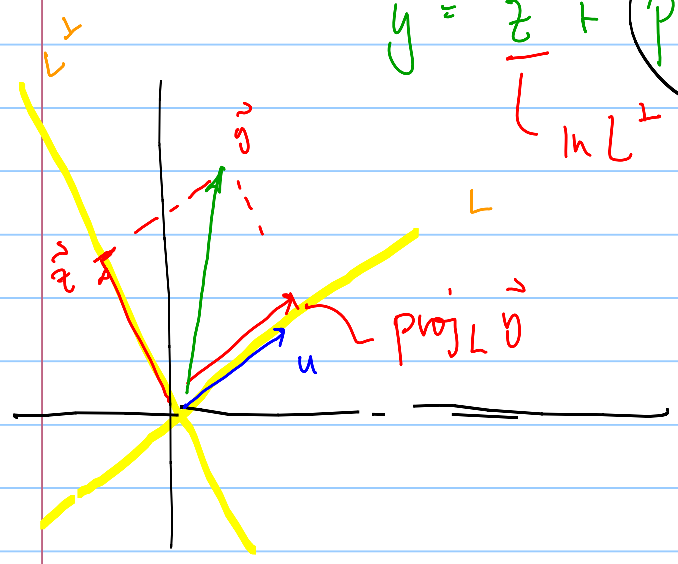
$\vec{y} \in \mathbb{R}^n \qquad \vec{u} \in \mathbb{R}^n. \qquad L = \text{Span}\{\vec{u}\}.$

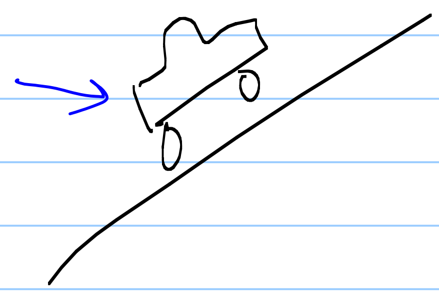
$$\text{proj}_L \vec{y} = \left[\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right]\vec{u}$$ $\rightarrow$ unchanged for any scalar mult of $\vec{u}$.

$\vec{z} = \vec{y} - \text{proj}_L \vec{y} \qquad \vec{z}$ is in $L^\perp$

$$\vec{y} = \underbrace{\vec{z}}_{\text{in } L^\perp} + \underbrace{\text{proj}_L \vec{y}}_{\text{in } L}$$

Same as $\hat{y}$ from last lecture + text.





(6.3) $\vec{y} \in \mathbb{R}^n$ W subspace of $\mathbb{R}^n$.

$$\vec{y} = \underbrace{\vec{z}}_{\in W^\perp} + \underbrace{\text{Proj}_W \vec{y}}_{\in W}.$$

$$\vec{z} = \vec{y} - \text{proj}_W \vec{y}.$$

Orthogonal Decomposition Theorem

W subspace $\mathbb{R}^n$ $\vec{y} \in \mathbb{R}^n$

$\vec{y}$ can written uniquely $\underbrace{\text{proj}_W \vec{y}}_{\in W} + \underbrace{\vec{z}}_{\in W^\perp}$

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ ~~orthogonal~~ orthonormal basis for W

$$\text{Proj}_W \vec{y} = \left[\frac{\vec{y}\cdot\vec{u}_1}{\cancel{\vec{u}_1\cdot\vec{u}_1}_{\,1}}\right]\vec{u}_1 + \cdots + \left[\frac{\vec{y}\cdot\vec{u}_p}{\cancel{\vec{u}_p\cdot\vec{u}_p}_{\,1}}\right]\vec{u}_p$$

$$\vec{z} = \vec{y} - \text{proj}_W \vec{y}.$$

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ $\text{Proj}_W \vec{y} = \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{z} = \vec{y} - \text{Proj}_W \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{9}{15} - 1 \\ \frac{9}{6} + \frac{1}{2} \\ -\frac{9}{30} + \frac{1}{2} \end{bmatrix}$$

$$* = \frac{\vec{u}_1 \cdot \vec{y}}{\vec{u}_1 \cdot \vec{u}_1} = \frac{2 \cdot 1 + 5 \cdot 2 + (-1)3}{2 \cdot 2 + 5 \cdot 5 + (-1)(-1)} = \frac{9}{30}$$

If $\vec{y} \in \mathbb{R}^n$ W subspace of $\mathbb{R}^n$.

$\text{proj}_W \vec{y}$ closest "match" to $\vec{y}$ of all vectors in W.

$$\underbrace{\|\vec{y} - \text{proj}_W \vec{y}\|}_{\text{error of approx.}} < \|\vec{y} - \vec{v}\|$$

for all $\vec{v} \in W$
$\vec{v} \neq \text{proj}_W \vec{y}$

Example: $\vec{y} = \begin{bmatrix} -1 \\ -5 \\ -10 \end{bmatrix}$ $\vec{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\text{proj}_W \vec{y} = \frac{-5 + 10 - 10}{25 + 4 + 1} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-1 - 10 + 10}{1 + 4 + 1} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\|\vec{y} - \text{proj}_W \vec{y}\| = \left\| \begin{bmatrix} 0 \\ -5 \\ -10 \end{bmatrix} \right\| = \sqrt{(-5)^2 + (-10)^2}$$

$$= \sqrt{125} = 5\sqrt{5}$$

An Orthonormal basis = orthogonal basis in which all the vectors are <u>normalized</u> to 1

$$\|\vec{u}_i\| = 1$$

Given $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ orthogonal basis

$$\vec{u}_i \leftarrow \frac{\vec{u}_i}{\|u_i\|} \rightsquigarrow \text{orthonormal basis.}$$

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad \vec{V}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{V}_3 = \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$$

$$\|\vec{V}_1\| = \sqrt{3^2+1^2+1^2} = \sqrt{11} \qquad \vec{u}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$$

$$\|\vec{V}_2\| = \sqrt{(-1)^2+2^2+1^2} = \sqrt{6} \qquad \vec{u}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

$$\|\vec{V}_3\| = \sqrt{(-1)^2+(-4)^2+7^2} = \sqrt{66} \qquad \vec{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

U $m \times n$ matrix w/ orthonormal columns.

$$U = \begin{array}{c c} & n \\ m & \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} \end{array} \qquad U^T = \begin{array}{c c} & m \\ n & \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{bmatrix} \end{array}$$

$$\underset{n \times m}{U^T} \cdot \underset{m \times n}{U} = \underset{n \times n}{I_n}$$

$$U^T \begin{bmatrix} \\ \\ \vec{u}_i \\ \\ \end{bmatrix} \begin{bmatrix} & & \\ & \vec{u}_j & \\ & & \end{bmatrix} = \begin{array}{c c} & j \\ i & \begin{bmatrix} & & \\ & \vec{u}_i \cdot \vec{u}_j & \\ & & \end{bmatrix} \end{array}$$

An $m \times n$ matrix U has orthonormal columns iff $U^T U = I_n$.