

Systems of linear equations:

$$\begin{aligned}x_1 - 3x_3 + 4x_4 &= 2 \\2x_1 + x_2 + 2x_3 - x_4 &= 0 \\-x_2 + 4x_3 + x_4 &= 1\end{aligned}$$

Translate into coefficient matrix

$$\begin{vmatrix} 1 & 0 & -3 & 4 \\ 2 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \end{vmatrix}$$

Augmented matrix

$$\begin{vmatrix} 1 & 0 & -3 & 4 & 2 \\ 2 & 1 & 2 & -1 & 0 \\ 0 & -1 & 4 & 1 & 1 \end{vmatrix}$$

Gaussian Elimination \rightarrow row echelon form.

already determines positions of pivots: pivot cols, rows
free variables, & whether system has a solution.

Can use to solve for soln by back substitution.

red row ech form

can read off solutions.

can use to find parametric description of solutions.

1.2. Vectors ($n \times 1$ matrix)



vector add, scalar mult.

linear combinations of vectors.

Sys of lin eqn as a vector eqn:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Span $\{\vec{v}_1, \dots, \vec{v}_p\}$ Set of all linear combinations of $\vec{v}_1 \dots \vec{v}_p$ (geometric view. For example $\text{Span}\{v_1, v_2\}$ in 3D is a plane thru the origin if lin indp.).

Matrix Equation

$$A\vec{x} \quad A = [\vec{a}_1 \dots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

only defined if # entries of $x = \# \text{ cols of } A$.

defined to be $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$

Rephrase System of linear equations as a matrix equation: $A\vec{x} = \vec{b}$.

A an $m \times n$ matrix. All equiv statements.

- 1) For every $\vec{b} \in \mathbb{R}^m$ $A\vec{x} = \vec{b}$ has a solution.
- 2) Every $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A .
- 3) Columns of A span \mathbb{R}^m
- 4) A has a pivot in every row.

Homogeneous System of Linear Eqs: $A\vec{x} = \vec{0}$

(helps characterize properties of A). $\vec{0}$ has a non-trivial solution iff A has a free variable.

Used rref(A) to find parametric vector form for the solutions to $A\vec{x} = \vec{0}$.

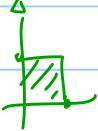
Lin Dep: $\vec{v}_1 \dots \vec{v}_p$ are lin dep if $\exists c_1 \dots c_p$ (not all 0) $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$.

If not lin dep, then lin indep.

A lin dep set of Vectors has some redundancy: can eliminate a vector and still have same span.

The cols of A are lin ^{indep} iff $A\vec{x} = \vec{0}$ has only the trivial solution.

Matrices as linear transformations: $T(\vec{x}) = A\vec{x}$. $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

 T is linear if $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$.
 $T(c\vec{u}) = cT(\vec{u})$

(Looked at examples
 $\exists T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
Has acted on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

T is linear $\Leftrightarrow \exists A \quad T(\vec{x}) = A\vec{x}$.

$$T \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_n) \end{bmatrix}$$

T is 1-1 iff $T(\vec{x}) = \vec{0}$ has only trivial solution. \rightarrow cols of A lin indep.
T is onto iff Columns of A span \mathbb{R}^m .

Ch 2

Matrices: A $m \times n$ matrix is n columns from \mathbb{R}^m .

Matrix add, scalar mult.

Matrix mult: $A \cdot B \quad \rightsquigarrow \begin{bmatrix} AB_1 & AB_2 & AB_3 \end{bmatrix}$
 $m \times n \quad n \times l$. in \mathbb{R}^m
mxl matrix.

$$A = [\vec{a}_1 \dots \vec{a}_n] \quad B = [\vec{b}_1 \dots \vec{b}_l].$$

$A \cdot B$ $m \times l$ $i \begin{bmatrix} \text{---} \end{bmatrix} \begin{bmatrix} | \\ | \end{bmatrix} = i \begin{bmatrix} \text{---} \\ \bullet \end{bmatrix}$
 $(m \text{ col } i \text{ of } A) \cdot (l \text{ row } j \text{ of } B)$

$$A \cdot B \quad AB\vec{x} = T(Q(\vec{x}))$$

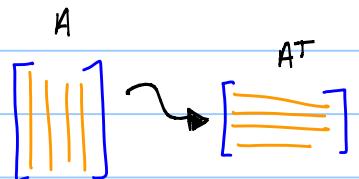
↓ ↓
 1 Q

In general $AB \neq BA$.

$$A(B+C) = AB + AC$$

Transpose: Rows of A^T = cols of A .

$$(AB)^T = B^T A^T$$



Inverse of an $n \times n$ matrix:

$$AA^{-1} = A^{-1}A = I.$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

inv. of A.

To compute A^{-1}

$$\left[\begin{array}{c|c} A & I \end{array} \right] \xrightarrow{\text{G.A.}} \left[\begin{array}{c|c} I & A^{-1} \end{array} \right]$$

If rref of $A + I$ then A is not invertible.

Invertible Matrix Theorem: Let A be a square ($n \times n$) matrix

A is invertible

A is row equiv to I

n pivot cols

n pivot rows

cols of A are lin indep.

$A\vec{x} = \vec{0}$ has only triv soln

Vector Spaces: "vectors" Vector addition.

\mathbb{R}^m

polynomials.

functions over subsh of \mathbb{R} .

zero vector, additive inverse.

Scalar mult

Scalar unity.

Subspace $W \subseteq V$

W : has zero vector

closed under scalar mult

and addition.

Examples: $\text{Span } \{\vec{v}_1, \dots, \vec{v}_p\} \quad \vec{v}_i \in \mathbb{R}^n$

Other Vector Spaces

If $m \times n$ matrix

$\text{Col } A = \text{span } \{\vec{a}_1, \dots, \vec{a}_n\} \subseteq \mathbb{R}^m$

$\text{Nul } A = \text{all } \vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}$.

($\text{Col } A = \mathbb{R}^m$ if there are m pivots.)

W subspace of V . A basis for W is: $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$

(1) \mathcal{B} is lin indep.

different description of W .

(2) $\text{Span } \mathcal{B} = W$.

Example: basis for P_n : $\{1, k, \dots, k^n\}$

H Subspace of V .

$\text{Span } \{\vec{v}_1, \dots, \vec{v}_p\} = H$.

If $\vec{v}_1 \dots \vec{v}_p$ is not lin indep, there is a \vec{v}_i that can be removed + span stays the same.

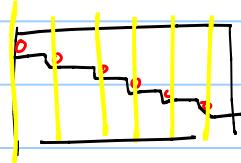
\Rightarrow Can keep removing vectors until you have a basis.

Given $m \times n$ A find a basis for $\text{Col } A$
 $\text{Nul } A$

$\text{Nul } A$: parametric description of solutions for $A\vec{x} = \vec{0}$
gives a basis for $\text{Nul } A$.

Col A:

$A \rightsquigarrow$ row echelon form of A:



indicates which columns are pivot cols. Go back to A & take those cols from A to get a basis for Col A.

4.4 Basis as Coordinate Systems.

$B = \{\vec{b}_1, \dots, \vec{b}_n\}$ basis for V.

$\vec{x} \in V$.

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad c_1 \vec{b}_1 + \dots + c_n \vec{b}_n = \vec{x}.$$

→ this is unique as long as B is a basis.

To find, solve for c_i :

$$B\vec{c} = \vec{x}$$

$$\vec{x} \begin{bmatrix} \vec{b}_1 & \dots & \vec{b}_n \end{bmatrix}$$

$\vec{x} \rightarrow [\vec{x}]_B$ is a bijection.

4.5 All bases for vector space V have the same # of elements (as long as there is a finite basis).
This is the dimension of V.

No finite basis → infinite dimensional.

(Any lin indep set can be expanded to become a basis).

4.6 Matrix Rank: A an $m \times n$ matrix

$$\dim \text{Row } A = \dim \text{Col } A = \# \text{ pivots} = \text{rank of } A.$$

$$\# \text{ non-pivot cols} = \# \text{ free vars} = \dim \text{Nul } A.$$

$$\text{rank } A + \dim \text{Nul } A = n. \quad (\text{max irr matrix}$$

$$\text{rank } A = n. \quad \text{Nul } A = \{ \vec{0} \}$$

$$\dim \text{Nul } A = n. \quad \text{etc.})$$

3.1 Determinants: ($n \times n$ matrices)

$$\text{Det of } 2 \times 2 \text{ matrix: } \text{Det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

For larger matrices: A_{ij} : matrix obtained by removing row i + col j from A .

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{n+1} a_{nn} \det A_{nn}.$$

$$\begin{bmatrix} & j \\ i & \end{bmatrix}$$

↑ pick a row or col that makes this easier - lots of 0's.

For triangular matrix, determinant is product of diagonal entries.

Sec 3.2

Row Ops & Determinants.

$$\begin{array}{ll} A \xrightarrow{\text{one row swap}} B & \det A = \det B \\ A \xrightarrow{\text{scalar mult by } k} B & k \cdot \det A = \det B \\ A \xrightarrow{\text{row swap}} B & \det A = -\det B. \end{array}$$

Can simplify A by applying row operations to get B .
Then find $\det B \xrightarrow{\text{from above}} \det A$.

(may need to keep track of scalar multiples + changes in sign)

Intuition behind \det : formula that evaluates to 0
when A is not invertible.

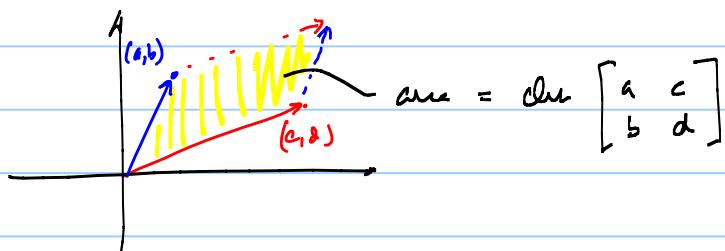
$$A \text{ invertible} \iff \det A \neq 0.$$

Determinant facts: $\det A = \det (A^T)$.

$$\det AB = \det A \cdot \det B.$$

3.3

geometric interpretation: \det of 2×2 matrix = area of parallelogram defined by cols.



in 3D: \det of matrix A is volume of parallelepiped defined by 3 cols of A .

5.1 Eigenvalues & Eigenvalues: $A \vec{v} = \lambda \vec{v}$

eigenvalue λ \uparrow eigenvector \vec{v}

If \vec{v} is an eigenvector of A , can find corresponding eigenvalue by multiplication: $A \vec{v} = \vec{w} \quad \vec{w} = \lambda \vec{v}$ for some λ .

Given matrix A & eigenvalue λ find corresponding eigenvectors(s).

$$\underbrace{(A - \lambda I)}_{=B}$$

$$\text{find } B\vec{x} = \vec{0}$$

If solution to this homogeneous eqn is an eigenvector corresponding to eigenvalue λ .

If λ is an eigenvalue of A , there will be at least one solution.

Method from Section 1.5 will give a basis of the eigenspace corresponding to eigenvalue λ .

If A is triangular, eigenvalues are diagonal entries.

5.2 To find eigenvalues if you do not have an eigenvector need to solve characteristic eqn.

$$\det [A - \lambda I] = 0$$

$\hookrightarrow \lambda$ is a variable here.

Need to solve for values of λ that make this eqn hold.

$\det(A - \lambda I)$ is a poly of degree n if A is an $n \times n$ matrix.

Once all roots found $\lambda_1, \dots, \lambda_k$

Can use these to find corresponding eigenvectors.

Two matrices A & B are similar if \exists invertible P s.t

$$A = PBP^{-1}$$

5.3 Diagonalization: If we can express A as PDP^{-1}

where D is diagonal & P is invertible then

cols of P are
eigenvectors of A

Diagonal elements of A are eigenvalues

$$A^k = P D^k P^{-1}$$

↳ easy to compute for diagonal matrices.

When is A diagonalizable? (See 5.3 Summary).

Solve for roots of char egn $\lambda_1 \dots \lambda_k$

m_1, \dots, m_k ← multiplicities.

$$\text{Want } (A - \lambda_i I) \vec{x} = \vec{0}$$

to have m_i lin indep solns.

If there are n distinct eigenvalues, then the matrix
is always diagonalizable.

6.1. Inner product of two vectors $\vec{w} \cdot \vec{v} = \vec{w}^T \vec{v}$

$$(w_1 v_1 + w_2 v_2 + \dots + w_n v_n)$$

$$[\vec{w}^T] \begin{bmatrix} \vec{v} \end{bmatrix}$$

$1 \times n \quad n \times 1 \rightarrow 1 \times 1$ scalar.

$$\text{Length of a vector } \|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \dots + v_n^2}$$

only 0 if $\vec{v} = \vec{0}$, otherwise $\|\vec{v}\| > 0$.

$$c\|\vec{v}\| = \|\vec{cv}\|$$

Allows us to define a notion of distance between two vectors:

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

$\vec{v} + \vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.

W Subspace of \mathbb{R}^n W^\perp is the set of all vectors orthogonal to everything in W .

W^\perp is also a vector space.

$$(\text{Row } A)^\perp = \text{Nul } A \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

~~6.2~~^{+6.3} Orthogonal Sets. $\{\vec{v}_1, \dots, \vec{v}_p\}$ each pair is orthogonal.

Orthogonal sets are lin indep.

Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ an orthogonal set.

If $\vec{y} \in \text{Span } \{\vec{u}_1, \dots, \vec{u}_p\}$ $\vec{y} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$$

$\{\vec{u}_1, \dots, \vec{u}_p\}$ is a basis B of $\text{Span } \{\vec{u}_1, \dots, \vec{u}_p\}$
 $[\vec{y}]_B$ is easier to compute if
 B is an orthogonal set.

Let W be a subspace of \mathbb{R}^n $y \in \mathbb{R}^n$. $\{\vec{u}_1, \dots, \vec{u}_p\}$ orthog bases for W .

$$\underline{\underline{\text{proj}}}_W \vec{y} = \left[\frac{\vec{u}_1 \cdot \vec{y}}{\vec{u}_1 \cdot \vec{u}_1} \right] \vec{u}_1 + \left[\frac{\vec{u}_2 \cdot \vec{y}}{\vec{u}_2 \cdot \vec{u}_2} \right] \vec{u}_2 + \dots + \left[\frac{\vec{u}_p \cdot \vec{y}}{\vec{u}_p \cdot \vec{u}_p} \right] \vec{u}_p.$$

this vector is in W and is the closest vector to \vec{y} among all the vectors in W .

$$\vec{y} - \text{proj}_W \vec{y}$$
 is orthogonal to W .

$$\vec{y} = \underbrace{\vec{z}}_{\in W} + \underbrace{\text{proj}_W \vec{y}}_{\in W^\perp}$$

= $\vec{y} - \text{proj}_W \vec{y}$ (in W^\perp).

If $\{\vec{v}_1 \dots \vec{v}_p\}$ is an orthonormal basis of W
 \hookrightarrow orthogonal + normalized to 1.

Let $V = [\vec{v}_1 \dots \vec{v}_p]$

then $\text{proj}_W \vec{y} = VV^T \vec{y}$.

If V has orthonormal columns then $V^T \cdot V = I$.

6.4 Given lin indep set $\{\vec{x}_1 \dots \vec{x}_p\}$

Find orthogonal $\{\vec{v}_1 \dots \vec{v}_p\}$ $\text{Span}\{\vec{x}_1 \dots \vec{x}_p\} = \text{Span}\{\vec{v}_1 \dots \vec{v}_p\}$.
 by Gram-Schmidt.

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

⋮

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}.$$

6.5 Least Sq If $A\vec{x} = \vec{b}$ not solvable, still want to find \vec{x} that gets as close to \vec{b} as possible.

$$\vec{b}' = \text{Proj}_{\text{Col } A} \vec{b}$$

Solve $A\vec{x} = \vec{b}'$
 this will also be a solution to:
 $A^T A \vec{x} = A^T \vec{b}'$.

7.1 Symmetric matrix: $A^T = A$.

If A symmetric then two eigenvectors w/ different eigenvalues are orthogonal.

A is symmetric $\iff A$ is diagonalizable $A = PDP^{-1}$
 cols of P form an orthonormal set.
 $A = P D P^T$
 (if cols of P are an orthonormal set
 then $P^{-1} = P^T$).