# Friday Oct 17

Note Title 10/17/2014

**Theorem** For every $n \geq 1$ $P(n)$ is true.

The principle of induction says it is sufficient to prove:

(1) $P(1)$ is true (base case)

(2) $(n \geq 1$ and $P(n)) \rightarrow P(n+1)$ (inductive step).

For the inductive step we say:

Assume: $n \geq 1$ and $P(n)$ is true.

Prove: $P(n+1)$ is true.

(2) The inductive step is the same as:

$(P(1) \rightarrow P(2)) \wedge (P(2) \rightarrow P(3)) \wedge (P(3) \rightarrow P(4)) \wedge \cdots$

Could also say for Inductive step:

Assume: $n \geq 2$ and $P(n-1)$ is true.

Prove: $P(n)$

**Theorem** For every $n \geq 1$, $\underbrace{3 \text{ evenly divides } 2^{2n} - 1}_{3k,\ k \text{ int.}}$.

$P(9)$ : 3 evenly divides $2^{2 \cdot 9} - 1 = 2^{18} - 1$.

3 evenly divides $m$ $\equiv$ $\underbrace{m = 3k \text{ for some int } k}$

Pf Base case: $n = 1$ 3 evenly divides $2^{2 \cdot 1} - 1 = 3$ ✓

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow \cdots$

Ind step: Assume $n \geq 2$ and 3 evenly divides $2^{2(n-1)} - 1$

Prove 3 evenly divides $2^{2n} - 1$.

Assume $2^{2(n-1)} - 1 = 3k$ for $k$ int.

$2^{2n-2} - 1$.

$$2^{2n} - 1 = 2^2 (2^{2n-2} - 1) + 3.$$

$2^2 (2^{2n-2} - 1)$

$2^{2n-2+2} - 2^2$

$2^{2n} - 4$

$$= 4(3k) + 3$$

$$= 3\underbrace{(4k+1)}_{\text{int}}$$

$\Rightarrow$ $2^{2n} - 1$ is a multiple of 3.

Theorem: for $n \geq 4$ $n! \geq 2^n$.

$n! \geq \_\_ = \_\_ \geq \_\_ \geq \cdots\cdots \geq 2^n$

If $A \geq B$ $\qquad \square + A \geq \square + B$

$\square + 2A \geq \square + 2B$

Proof: Base case $n=4$ $\quad 4! \geq 2^4$

$24 \geq 16$ ✓

Ind Step: Assume $n \geq 4$ $\quad n! \geq 2^n$ $\qquad n>4 \quad (n-1)! \geq 2^{n-1}$

Prove $\quad (n+1)! \geq 2^{n+1}$ $\qquad n! \geq 2^n$

$$(n+1)! = (n+1)\underbrace{n!} \geq (n+1)2^n \geq 2 \cdot 2^n = 2^{n+1}.$$

$n+1 \geq 2$, because $n \geq 4$.

Theorem For $n \geq 4$ $\quad n^2 \geq 3n+4$.

Pf Base $n=4$ $\quad 4^2 \geq 3 \cdot 4 + 4$

$16 \geq 16$

Ind Step. Assume $n \geq 4$ $\quad n^2 \geq 3n+4$ $\qquad 3n+7.$

Prove $\quad (n+1)^2 \geq 3(n+1)+4$

By the I.H.

$$(n+1)^2 = \underbrace{n^2}+2n+1 \geq (3n+4)+2n+1$$

$$= 5n+5$$

$$= 3n+2n+5$$

If $n \geq 4$ then $n \geq 1$.

$n \geq 4$

~~$n \geq 1$~~.

$$\geq 3n+2 \cdot 4+5$$

$$= 3n+13 \geq 3n+7.$$

Fib :

$$\boxed{\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2}. \end{aligned}}$$

Thrm: for $\{f_n\}$ defined above:

$$f_n \le 2^n.$$

Pf Base cases: for $n=0$ $f_0 = 1$ (by def)

$2^0 = 1$

$f_0 \le 2^0$

$n=1$ $f_1 = 1$ (by def)

$2^1 = 2$

$f_1 \le 2^1$

Assume $n \ge 2$. $f_{n-1} \le 2^{n-1}$

Prove $f_n \le 2^n$.

$f_n = f_{n-1} + f_{n-2}$ by def.

Strong Induction: Assume $P(1) \wedge P(2) \wedge \cdots P(n-1)$

Prove $P(n)$.

Weak

$P(n-1) \rightarrow P(n)$

Strong

$\left(P(1) \wedge P(2) \wedge \cdots P(n-1)\right] \Rightarrow P(n).$

$f_0 \le 2^0$ and $f_1 \le 2^1$

Assume for $k = \underline{0, 1, \ldots, n-1}$ $\quad f_k \le 2^k$

Prove $f_n \le 2^n$.

$P(0) \wedge P(1) \wedge P(2) \wedge P(3)$ — — —

$P(0) \wedge P(1) \rightarrow P(2)$

$P(0) \wedge P(1) \wedge P(2) \rightarrow P(3)$

By strong ind.

$f_{n-1} \le 2^{n-1}$

$f_{n-2} \le 2^{n-2}$

$$\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
&\le 2^{n-1} + 1 \cdot 2^{n-2} \\
&\le 2^{n-1} + 2 \cdot 2^{n-2} \\
&= 2^{n-1} + 2^{n-1} = \\
&= 2 \cdot 2^{n-1} = 2^n \quad \boxtimes
\end{aligned}$$

$\le 2^n$