Note Title 11/5/2014

Bijection Rule:

If A & B are finite sets and there is a bijection between A & B. ($f: A \to B$ where f is a bijection) then $|A| = |B|$.

1 to 1
onto.

E = Set of 5-bit strings w/ an even # of 1's.

10110 $\notin E$
10111 $\in E$.
00000 $\in E$.

10111
10010

Bijection between $f: \{0,1\}^4 \to E$.

$$\boxed{|E| = |\{0,1\}^4| = 2^4}$$

$f(x) =$ $x0$ if x has an even # of 1's
$x \in \{0,1\}^4$
$x1$ if x has an odd # of 1's.

$x, x' \in \{0,1\}^4$ $\quad x \neq x'$ then $f(x) \neq f(x')$ (1-1)

if $y \in E$ $\quad y = \underline{b_1 b_2 b_3 b_4} b_5$

$b_5 = 1$ if $b_1 - b_4$ has an odd # of 1's.
$b_5 = 0$ if $b_1 - b_4$ has an even # of 1's.

$\Rightarrow f(b_1 b_2 b_3 b_4) = b_1 b_2 b_3 b_4 b_5 = y$.

onto.

Permutations.

order matters

A permutation of a finite set S is a sequence of the elements in S such that all items appear exactly once.

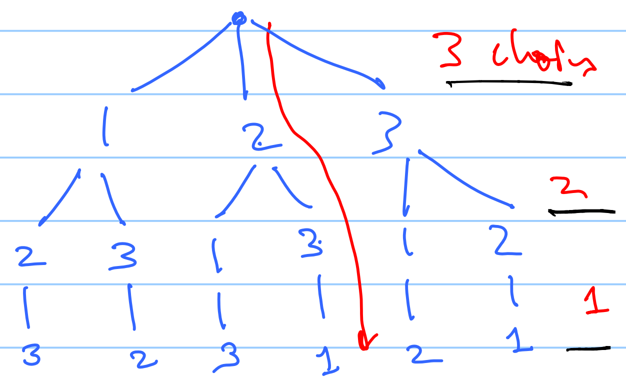
Example $S = \{1, 2, 3\}$.

List permutations.

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

$\#$ permutations of S.

Decision tree for permutations.



$3 \cdot 2 \cdot 1 = 6$

$S = \{1, 2, \ldots, n\}$

Decision tree approach.

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

$\#$ choices for 1^{st} element

$\#$ choices for 2^{nd} element

$\vdots$

$\#$ permutations of a set with n elements $= n!$

Generalized Product Rule

Consider a set of sequences of k items.

n_1 choices for first item

For every possible choice for the 1^{st} item, there are n_2 possible choices for the 2^{nd} item

For every possible choice for the first two items, there are n_3 possible choices for the 3^{rd} item

$\vdots$

For every possible choice for the first $k-1$ items, there are n_k possible choices for the k^{th} item

possible sequences is: $n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_k$.

The set of possible choices may depend on previous choices but the number of possible choices does not depend on previous choices.

Restaurant Selection:

There are 10 different lunch specials.

Diner selects a different entrée each day of the work week (M-F).

How many different selections possible?

↳ it matters which choices is made on which day.

$$\frac{10!}{5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$\underset{M}{\underline{10}} \cdot \underset{T}{\underline{9}} \cdot \underset{W}{\underline{8}} \cdot \underset{Th}{\underline{7}} \cdot \underset{F}{\underline{6}} = \frac{10!}{5!}$$

What if there are 7 drink choices and the person does not repeat a drink selection?

$$(10\cdot 9\cdot 8\cdot 7\cdot 6)(7\cdot 6\cdot 5\cdot 4\cdot 3) = \frac{10!}{5!}\cdot\frac{7!}{2!}$$

$$\underset{M}{\underline{10\cdot 7}} \cdot \underset{T}{\underline{9\cdot 6}} \cdot \underset{W}{\underline{8\cdot 5}} \cdot \underset{Th}{\underline{7\cdot 4}} \cdot \underset{F}{\underline{6\cdot 3}}$$

What if drink repetitions are allowed?

Entrée

$$\frac{10!}{5!} \quad 7^5$$

Teacher has four different jobs to fill:

Cutting 20
Stapling 19
Glueing 18

$20 \cdot 19 \cdot 18 = \frac{20!}{17!}$

20 kids in the class and no kid gets more than one job.

An r-permutation over a set of n elements is a sequence of r elements from the set with no repetitions. $(r \le n)$.

Examples of 3-~~repetitions~~ permutations from $\{1, 2, 3, 4, 5\}$.

$(4, 1, 3)$
$(1, 4, 3)$

# of 3-~~repetitions~~ perms from $\{1, 2, 3, 4, 5\}$.

$$\underline{5} \cdot 4 \cdot 3 = \frac{5!}{2!}$$

The # of r-~~repetitions~~ perms. from a set of n objects is
$$P(n, r) = \underset{(n-0)}{n} \cdot (n-1) \cdots \underset{(n-(r-1))}{\cdot (n-r+1)} = \boxed{\frac{n!}{(n-r)!}}$$

$$\frac{n(n-1)\cdots(n-r+1)\cancel{(n-r)}\cancel{-}\cdots\cancel{1}}{\cancel{(n-r)}\cancel{-}\cdots\cancel{1}} = \frac{n!}{(n-r)!}$$

There are 20 computers in a distributed network.
There are 10 different jobs to assign to computers.
At most one job per computer.

How many ways to assign jobs to computers?

$$\underset{\text{Job 1}}{\underline{20}} \cdot \underset{\text{Job 2}}{\underline{19}} \cdot \underline{18} \; \cdots \; \_ \; \_ \; \_ \cdot \underset{\text{Job 10}}{\underline{11}} = \frac{20!}{10!}$$

$P(20,10)$

How many permutations of $\{1,2,3,4,5\}$ in which 1 is next to 2?

5 4 1 2 3

3 4 5 2 1

~~3 1 5 2 4~~

Case 1

1 right of 2 — # perms

2-1

2-1, 3, 4, 5 — $4!$

Case 2 — 1 left of 2

1-2, 3, 4, 5 — $4!$

$$(4!+4!) = 2\cdot 4!$$

Counting Subsets.

Committee of 5 $\{$Abigail, Benjamin, Charlie, Daniel, Eleanor$\}$

ways to Select a Pres, Secretary, Treasurer.

($\underset{\text{Pres}}{\underline{\text{Charlie}}}$, $\underset{\text{Treasurer}}{\underline{\text{Abigail}}}$, $\underset{\text{Secretary}}{\underline{\text{Eleanor}}}$)

3-permutations from a set of 5. $= P(\underline{5,3}) = \frac{5!}{2!}$

Now only want to select a sub-committee of three. (No special roles).

$\{$ Charlie, Abigail, Daniel $\}$

$= \{A, C, D\}$

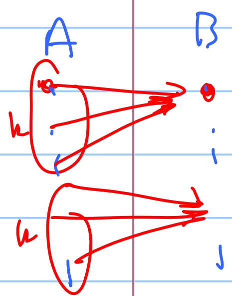
$= \{D, C, A\}$

(A, C, D) –
(A, D, C) –
(C, A, D) –
(C, D, A) –
(D, A, C) –
(D, C, A) –

Every subset of 3 corresponds to 3! 3-permutations.

different subsets of size 3 from a set of size 5 $= \frac{P(5,3)}{3!}$

k-to-1 rule: A & B finite sets.



$$f : A \to B.$$

For every $y \in B$ there are exactly k different $x_1, x_2, \ldots, x_k \in A$ $\quad f(x_i) = y$. $\Big\}$ f is k-to-1

Then $\quad |A| = k\,|B| \qquad |B| = \dfrac{|A|}{k}.$

f : 3-permutations of $\{A,B,C,D,E\} \longrightarrow$ 3-subsets of $\{A,B,C,D,E\}$.

$f(A,B,C) = \{A,B,C\}$

$f(A,C,B) = \{A,B,C\}$ $\qquad$ f is $3!$ to 1

$\vdots$

$f(C,B,A) = \{A,B,C\}$ $\qquad$ # 3-subsets = $\dfrac{\text{\# 3-perm}}{3!}$

An r-subset of a set S is a subset $A \subseteq S$
$|A| = r$.

$r \le n$

The # of r-subsets of a set of n elements is

$$\frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r} = C(n,r)$$

"n choose r"

$S = \{1, 2, \ldots, n\}$.

$$f(\underline{i_1, i_2, \ldots, i_r}) = \{i_1, i_2, \ldots, i_r\}.$$

any permutation of these maps to the same subset.

f is $r!$-to-1.

$$\#\text{ r-subsets of } \{1, \ldots, n\} = \frac{P(n,r)}{r!}$$

Show: $\binom{n}{r} = \binom{n}{n-r}$