ways to select a subset of k elements from a set of size n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}$$

Example $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

4-subset

$\{2, 4, 8, 9\}$

4 8 9 2

4-permutation

$(2, 4, 8, 9)$

$(9, 4, 2, 8)$

$\vdots$

$4!$-to-1 correspondence

Fact: $\binom{n}{r} = \binom{n}{n-r}$

By arithmetic.

$$\frac{n!}{r!\,(n-r)!}$$

$$\frac{n!}{(n-r)!\,\underbrace{(n-(n-r))!}_{r!}}$$

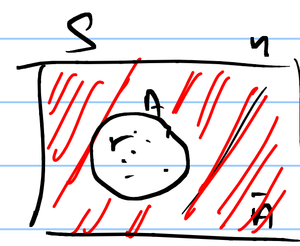
By bijection: $S = \{1, 2, \ldots, n\}$

A subset of r items from S can be specified by indicating which $n-r$ are not selected.

Sub_r = set of subsets of S w/ r elements.
Sub_{n-r} = set of subsets of S w/ $n-r$ elements.

Know. $|Sub_r| = \binom{n}{r}$

Know. $|Sub_{n-r}| = \binom{n}{n-r}$



$f : Sub_r \to Sub_{n-r}$ $\quad f(A) = \bar{A}$ $\quad f(\underset{\bar{B}}{?}) = B$ ↳ size $n-r$.

$A \subseteq S$
$|A| = r$

[Exercise: show f is 1-1 and onto.]

How many binary strings of length 9 have exactly 4 1's?

0 0 1 1 0 0 0 1 1

0 0 1 0 1 0 1 1 0

$\underset{}{\underline{0}}\ \underset{4}{\underline{1}}\ \underline{0}\ \underset{4}{\underline{1}}\ \underset{4}{\underline{1}}\ \underline{0}\ \underline{0}\ \underline{0}\ \underset{4}{\underline{1}}$

# of ways to pick 4 positions for the 1's: $\binom{9}{4}$

By bijection $S = \{1,2,3,4,5,6,7,8,9\}$.

$A \subseteq S$ and $|A| = 4$

g: Subsets of S w/ 4 elements $\longrightarrow$ binary strings of length 9 w/ exactly four 1's.

$g(A) = $ if $j \in A$ j^{th} bit is 1
if $j \notin A$ j^{th} bit is 0.

$g(\{1,3,4,9\}) = \overset{1\,2\,3\,4\,5\,6\,7\,8\,9}{101100001}$

$g(A) = \underset{1\,2\,3\,4\,5\,6\,7\,8\,9}{011001010}$, what is A? $\longrightarrow \{2,3,6,8\}$.

Strings over the character set $\{0,1,2,3,4,5,6,7,8,9\}$

How many strings of length 8? 10^8

$\underset{10}{\overset{\uparrow}{\_}} \cdot \underset{10}{\overset{\uparrow}{\_}} \cdot \_ \ \_ \ \_ \ \_ \ \_ \cdot \underset{10}{\overset{\uparrow}{\_}}$

How many strings of length 8 have exactly three 5's?

$\underset{1}{\_} \ \underset{2}{5} \ \underset{3}{5} \ \underset{4}{\_} \ \underset{5}{\_} \ \underset{6}{\_} \ \underset{7}{5} \ \underset{8}{\_}$

9 choices $\leftarrow 9 \quad \leftarrow 9 \quad \leftarrow 9 \quad \leftarrow 9$

pick places for 5's $\binom{8}{3}$

# of ways to fill in the remaining places: 9^5

$$\boxed{\binom{8}{3} \cdot 9^5}$$

How many strings of length 8 have exactly 3 5's and 2 1's.

$5 \ 5 \ \_ \ 1 \ 5 \ 1 \ \_ \ \_$

$\binom{8}{3} \cdot \binom{5}{2} \cdot 8^3$

Distribution Problems

175 people in class.

"indistinguishable"

- # ways to distribute 10 identical prizes with at most one to each person?

people are distinct
distinguishable.

$$\binom{175}{10}$$

ways to distribute 10 different prizes with at most one to each person?

$$\underset{P1}{175} \cdot \underset{P2}{174} \ \underset{P3}{\_\_} \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \cdot \underset{P10}{166}$$

$P(175, 10)$

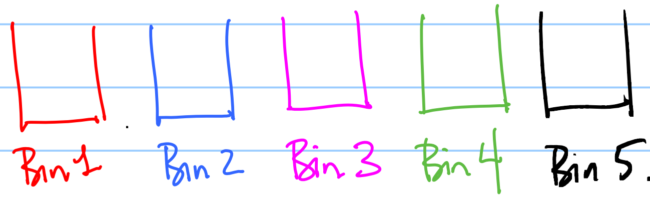
ways to distribute 10 different prizes with no limit on the # an individual can receive?

$$\underset{P1}{175} \cdot \underset{P2}{175} \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \ \_\_ \cdots \underset{P10}{175}$$

$\hookrightarrow (175)^{10}$.

Balls into bins

5 distinguishable bins.



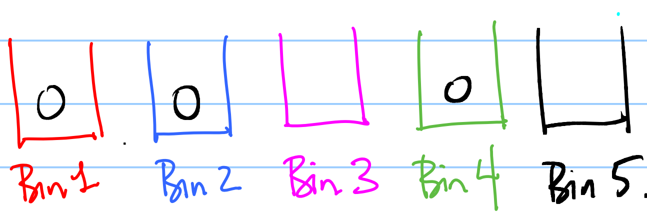
$m = \#$ bins
$n = \#$ balls.

Count # ways to place 3 balls if

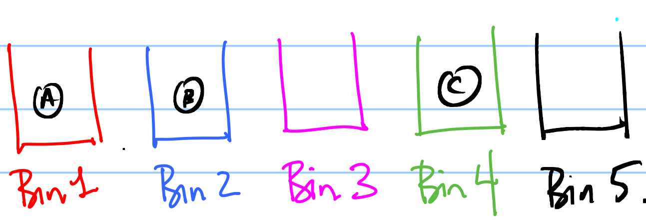
	Distinguishable Balls	Indistinguishable Balls.
≤ 1 per bin	$P(5,3)$ $P(m,n)$	$\binom{5}{3}$ $\binom{m}{n}$
No limit # in a bin	5^3 m^n	X

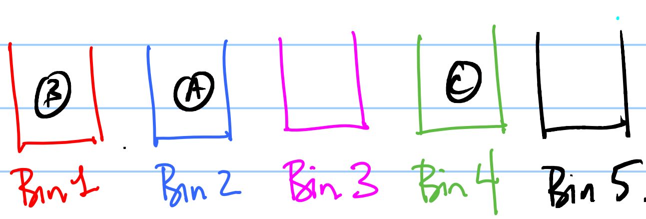
$m = \#$ bins.
$n = \#$ balls.

Indistinguishable Balls



Distinguishable Balls





Playing Card Examples.

Standard deck: 52 cards.

13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K.

4 suits: ♥ ♦ ♣ ♠

Each card has a rank + a suit: 8♡ Q♢

Every rank-suit combination possible:

$$\#\underline{\underline{\text{cards}}} = |\text{Suits}| \cdot |\text{Ranks}| = 4 \cdot 13 = 52.$$

5-card hand is a subset of 5-cards.

$\{$ A♢, A♠, 4♣, 2♢, 8♡ $\}$ — order doesn't matter.

- How many different 5-card hands? $\binom{52}{5}$

- How many five-card hands have:

  * exactly 3 clubs? — 13 clubs, 39 non-clubs.

ways to pick clubs $\rightarrow \binom{13}{3}\binom{39}{2} \rightarrow$ ways to pick non-clubs

8♥ 8♦ 8♣ K♦ 2♣

* 3-of-a-kind?

$13 \cdot \binom{4}{3} \cdot 48 \cdot 44$

$\binom{4}{3}$ → # of ways to pick cards in 3 of a kind.

13 → # ways to pick rank for the 3-of-a kind.

* 2 pairs. K♥ K♠ 3♦ 3♣ 8♠

$\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot 44$

$\binom{13}{2}$ ↳ ways to pick ranks for pairs.

Full House
3 of a kind
2 of a kind.

$13 \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$

* no face cards (no J, Q, K).

* ≥ 1 club.

A club has 10 men and 9 women.

How many ways to select a committee of 6?

* no restrictions.

* Same # men as women.

* Deloris & Frank do not both serve on the committee?

* ≥ 1 woman.

* ≤ 1 woman.