Note Title 11/21/2014

Fibonacci Sequence: $f_0 = f_1 = 1.$

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2.$

What is f_{1000}?

What is the smallest n such that $f_n \geq 5000$?

Solving a recurrence relation:

Given: Recurrence Relation + Initial Values.

Find "Closed form" solution.

Express f_n as a function of n, not earlier terms in the sequence.

For example: $f_0 = 7$

$f_n = f_{n-1} + 3$ for $n \geq 1.$

$f_n = 7 + 3 \cdot n$

$f_0 = 10$

$f_n = 4 \cdot f_{n-1}$ for $n \geq 1$

$f_n = 10 \cdot 4^n$

Linear Homogeneous Recurrence Relations.

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}$$

$c_1 \cdots c_k$ Constant.

If $c_k \neq 0$ the degree of the recurrence relation $= k$.

"homogeneous" means no extra terms.

		Linear?	Homogeneous?
$g_n = 3 \cdot g_{n-1} - 4 g_{n-2}$.	$d = 2$	Y	Y
$g_n = 4 g_{n-1} + \frac{g_{n-2}}{5}$	$d = 2$	Y	Y
$g_n = 4 g_{n-1} + 3$	$d = 1$	Y	N.
$g_n = \sqrt{2} g_{n-1} + 5 g_{n-7}$.	$d = 7$	Y	Y
$g_n = n \cdot g_{n-1} + g_{n-2}$.		N	Y.
$g_n = g_{n-1} \cdot g_{n-2} + g_{n-3}$.		N	Y

Solution to linear homogeneous recurrence relation.

"Guess" $f_n = x^n$.

$\Rightarrow$ $f_n = f_{n-1} + 2 f_{n-2}$.

• 2, −1.
• Characteristic eqn.
• General form

$\left(x^n = x^{n-1} + 2x^{n-2} \right) / x^{n-2}$

characteristic equation $x^2 = x^1 + 2$

$\Rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$.

$x = 2$ or $x = -1$.

$f_n = 2^n$ $\quad f_n = (-1)^n$

$\boxed{f_n = \alpha_1 2^n + \alpha_2 (-1)^n}$ general form

$a_n = 5a_{n-1} - 6a_{n-2}$. $\qquad a_n = x^n$

$x^n = 5x^{n-1} - 6x^{n-2}$

$\Downarrow$

$x^2 = 5x - 6$

$x^2 - 5x + 6 = 0 \quad \longrightarrow \quad (x-2)(x-3) = 0$

(characteristic eqn)

$a_n = \alpha_1 2^n + \alpha_2 3^n$

$g_n = 4g_{n-1} - g_{n-2} - 6g_{n-3}$. $\qquad 2, -1, 3.$

$x^n = 4x^{n-1} - x^{n-2} - 6x^{n-3}$

$(x-2)(x+1)(x-3) = 0$.

$x^3 = 4x^2 - x - 6$

$g_n = \alpha_1 2^n + \alpha_2(-1)^n + \alpha_3(3)^n$

$x^3 - 4x^2 + x + 6 = 0$.

Recurrence relation by itself has an infinite # of solutions.

$\Rightarrow$ Narrow down by initial conditions.

$a_n = a_{n-1} + 2a_{n-2} \rightsquigarrow a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

$a_0 = 2 \Rightarrow$

$a_1 = 7$

(1) $a_0 = 2 = \alpha_1 \cdot 2^0 + \alpha_2 (-1)^0$

(2) $a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 (-1)$

(1) $2 = \alpha_1 + \alpha_2$

(2) $7 = 2\alpha_1 - \alpha_2$

$9 = 3\alpha_1$

$\alpha_1 = 3$

$\alpha_2 = -1$

$a_n = 3 \cdot 2^n + (-1)(-1)^n$

$= 3 \cdot 2^n + (-1)^{n+1}$

$g_n = 4g_{n-1} - g_{n-2} - 6g_{n-3} \rightsquigarrow$

$\Rightarrow g_n = \alpha_1 2^n + \alpha_2 (-1)^n + \alpha_3 3^n$

$f_0 = 3$

$f_1 = -5$

$f_2 = -7$

- Degree 3 characteristic eqn. (3 distinct roots)
- $\Rightarrow$ General solution is a linear combination of 3 terms.
- 3 variable so solve for.
- 3 initial conditions $\Rightarrow$ 3 equations

If rec. relation is degree k

$\Rightarrow$ characteristic eqn is $p(x) = 0$ p is a degree k poly.

$\Rightarrow$ If p has distinct roots general solution has k variables

$\Rightarrow$ k initial conditions $\Rightarrow$ k equations

Solve for α_1 α_2 α_3 in g_n.

$f_0 = 3$

$f_1 = -5$

$f_2 = -7$

$$\Rightarrow g_n = \alpha_1 2^n + \alpha_2 (-1)^n + \alpha_3 3^n$$

$n=0,\ f_0 = 3 = \alpha_1 + \alpha_2 + \alpha_3$

$f_1 = -5 = 2\alpha_1 - \alpha_2 + 3\alpha_3$

$f_2 = -7 = 4\alpha_1 + \alpha_2 + 9\alpha_3$

$b_n = \underline{6 b_{n-1} - 9 b_{n-2}}$. $\quad a_0 = 1$, $a_1 = 6$. verify general solution.

$$x^n = 6x^{n-1} - 9x^{n-2}$$

$$(x-3)^2 = 0$$

$$x^2 = 6x - 9$$

$$x^2 - 6x + 9 = 0$$

$$b_n = 3^n$$

$$\underline{b_n = n \cdot 3^n}$$

$$n \cdot 3^n = 6 \cdot (n-1) \cdot 3^{n-1} - 9(n-2)3^{n-2} \qquad b_n = \underline{6 b_{n-1} - 9 b_{n-2}}.$$

$$n \cdot 3^n = 6 \cdot n \cdot 3^{n-1} - 6 \cdot 3^{n-1} - 9n\, 3^{n-2} + 18\, 3^{n-2}$$

$$2n3^n - \cancel{2 \cdot 3^n} - n3^n + \cancel{2 \cdot 3^n}$$

$$n \cdot 3^n = n 3^n$$

$$b_n = \alpha_1 \cdot 3^n + \alpha_2 n \cdot 3^n$$

$a_0 = 1$

$a_1 = 6$.

$n=0: \quad a_0 = 1 = \alpha_1$

$n=1: \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$.

Suppose char eqn for $\{a_n\}$ is.

$$(x-3)(x-5)^2(x+2)^3 = 0.$$

$$\quad 3 \qquad\qquad\qquad -2$$

Give general solution to a_n:

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 5^n + \alpha_3 n 5^n + \alpha_4 (-2)^n + \alpha_5 n(-2)^n + \alpha_6 n^2 (-2)^n$$

$$g_n = x^n$$

$$g_n = g_{n-1} + 8g_{n-2} - 12g_{n-3}. \qquad (x-2)^2(x+3).$$

$g_0 = 0$
$g_1 = 9$
$g_2 = 11,$

$$x^n = x^{n-1} + 8x^{n-2} - 12x^{n-3}$$

$$x^3 = x^2 + 8x - 12$$

$$x^3 - x^2 - 8x + 12 = 0.$$

$$(x-2)^2(x+3) = 0.$$

$$g_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot n2^n + \alpha_3(-3)^n$$

$n=0 \quad 0 = \alpha_1 + \alpha_3$ $\quad -2*$

$n=1 \quad 9 = 2\alpha_1 + 2\alpha_2 - 3\alpha_3$ $\quad -4*$

$n=2 \quad 11 = 4\alpha_1 + 8\alpha_2 + 9\alpha_3$

$g_0 = 0$
$g_1 = 9$
$g_2 = 11,$

$$9 = 2\alpha_2 - 5\alpha_3$$
$$11 = 8\alpha_2 + 5\alpha_3$$

$$20 = 10\alpha_2$$

$\alpha_2 = 2 \qquad \alpha_3 - 1$

$\alpha_1 = 1$

$$0 = \alpha_1 + \alpha_3$$
$$5 \quad \cancel{9} = 2\alpha_1 + \cancel{4} - 3\alpha_3$$

$$5 = -5\alpha_3$$