Generating Functions.

Note Title

$\boxed{\text{Office Hour Change}} \to$ No off hr SI
Tue 2-3 (11/25)
Moved to Wed 2-3 (11/26)

11/24/2014

Way of representing a sequence with an algebraic function.

Sequence: $\Rightarrow f_0, f_1, f_2, \ldots$

Generating function: $F(x) = f_0 + f_1 x + f_2 x^2 + \ldots$

x^k is a placeholder for f_k.

Terms in sequence become coefficients in the generating function.

Example: $\underset{f_0}{2}, \underset{f_1}{3}, \underset{f_2}{-7}, \underset{f_3}{1}, \underset{f_4}{5} \Rightarrow 2+3x-7x^2+x^3+5x^4 = F(x)$

$1, 1, 1, 1, \ldots$ $H(x) = 1+x+x^2+x^3+\ldots$

$g(x) = 3 \overset{+0x}{+} x^2 + 4x^3 - 10x^4 + 2x^6$ what is g_3? 4

$\{g_k\}$ $g_0 = 3$ $g_8 = 0$ $g_1 = 0$.

Compact representation of $f(x) = 1+x+x^2+\ldots$

only for $|x| < 1$.

$$\begin{aligned} H(x) &= 1 + x + x^2 + \ldots \\ - \quad xH(x) &= \phantom{1 +{}} x + x^2 + \ldots \\ \hline H(x) - xH(x) &= 1 \\ H(x) &= \frac{1}{1-x}. \end{aligned}$$

$\Big\}$ represents $1, 1, 1, \ldots$

What about the finite sequence 1, 1, 1, 1, 1

$F(x)$ $\underline{1 + x + x^2 + x^3 + x^4}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \ldots$$

$$\frac{x^5}{1-x} = \underbrace{x^5 + x^6 + \ldots}$$

$$F(x) = \frac{1}{1-x} - \frac{x^5}{1-x} = \frac{1-x^5}{1-x}$$

$$1 + x + \ldots + x^{20} = \frac{1-x^{21}}{1-x}$$

Sequence: $1, 0, 0, 0, \underset{x^4}{1}, 0, 0, 0, \underset{x^8}{1}, 0, \ldots$

$1 + x^4 + x^8 + x^{12} + \ldots$

$y = x^4$

$1 + y + y^2 + \ldots$

$$\frac{1}{1-y} = \frac{1}{1-x^4}$$

Generating functions can be used to represent the # of ways to select a subset of items from a pool of items. $\to$ S_k = # of ways to select k.

Examples:

- 2 apples, 1 banana. $\{A, A, B\}$

Select 0	Select 1	Select 2	Select 3
1	2	2	1
		$\{A, A\}$ $\{A, B\}$	

$S_0 = 1$, $S_1 = 2$, $S_2 = 2$, $S_3 = 1$

$S(x) = 1 + 2x + 2x^2 + x^3$

- Infinite supply of apples. $\to$ $A(x) = \frac{1}{1-x}$.

$a_0 = 1$, $a_1 = 1$, $a_2 = 1$, ...

- Infinite supply of 2 kinds of soda.

$S_0 = 1 \qquad S_1 = 2 \qquad S_2 = 3 \qquad S_3 = 4 \ldots$

$$S(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

- 5 apples.

$b_0 = 1$
$b_1 = 1$
$b_2 = 1$
$b_3 = 1$
$b_4 = 1$
$b_5 = 1$

$$B(x) = \frac{1 - x^6}{1 - x}.$$

$$\begin{aligned} S(x) &= 1 + 2x + 3x^2 + 4x^3 \ldots \\ x\,S(x) &= \phantom{1 +{}} x + 2x^2 + 3x^3 + \ldots \\ \hline \cancel{(1-x)}\,S(x) &= 1 + x + x^2 + \ldots = \frac{1}{\cancel{(1-x)}\,1-x} \end{aligned}$$

$$S(x) = \frac{1}{(1-x)^2}$$