Random Variables & Expectation.

Note Title 12/5/2014

A random variable is a way to summarize a quantity of interest in an outcome of an experiment.

Experiment with Sample Space S.

Random variable X is a function from S to $\mathbb{R}$.

$$X : S \to \mathbb{R}.$$

The range of X is denoted $X(S)$.

Example: experiment: 5 card hand.

C = # clubs in the hand.

$C(\{A\spadesuit, A\clubsuit, K\heartsuit, 2\clubsuit, 3\diamondsuit\}) = 2.$

What is the range of C? $\{0,1,2,3,4,5\}$

R = # 8's.

range of R $\{0,1,2,3,4\}$.

Example: Roll blue & red dice.

$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

Blue outcome, Red outcome.

M = largest of two numbers $M(x,y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x < y. \end{cases}$

range of M = $\{1,2,3,4,5,6\}$

$M(1,6) = 6$

$M(2,1) = 2.$

What is $M(4,2) = 4$

$M(5,6) = 6.$

What is $M(S)$? $\{1,\dots,6\}$

D = blue die − red die.

$D(5,2) = 5-2 = 3$

$D(3,4) = 3-4 = -1$

What is range of D? $\{-5,-4,\dots,4,5\}$.

Can define an event that a random variable takes on a particular value.

$[X=r] = \{s \in S : X(s) = r\}$.

What is the event that $D = -2$

$\hookrightarrow \{(1,3),(2,4),(3,5),(4,6)\}$.

What is $P[D=-2]$ (unif dist) $\frac{4}{36} = \frac{1}{9}$.

Back to M (max of red + blue die).

$[M = 1]$: $\{(1,1)\}$

$[M = 5]$: $\{(1,5), (2,5), (3,5), (4,5), (5,5),$
$(5,1), (5,2), (5,3), (5,4)\}$.

The distribution over a random variable is a complete description of that r.v. :

Set of ordered pairs : for each $r \in X(S)$ $(r, P[X=r])$.

Distribution over M: Range $\{1, \ldots, 6\}$.

Distribution over M: $2(r-1)+1$.

$(1, 1/36), (2, 3/36), (3, 5/36), (4, 7/36),$
$(5, \cancel{1}9/4), (6, 11/36)\}$.

The expectation of a random variable is the "average" value.

$$E[X] = \sum_{s \in S} X(s) \cdot p(s) = \sum_{r \in X(S)} r \cdot p[X=r].$$

$$\sum_{r \in X(S)} r \cdot \sum_{s:\, X(s)=r} p(s)$$

Experiment 2 tosses of a fair coin.

$$S = \{HH, HT, TH, TT\}.$$

random variable D: # of heads.

$$D(S) = \{0, 1, 2\}.$$

$p(s) = \frac{1}{4}.$

$s \in S$	HH	HT	TH	TT.
$D(s)$	2	1	1	0

$$E[D] = 2 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{4} + 0 \cdot \tfrac{1}{4} = 1$$

or...

r	0	1	2
Event $[D=r]$	$\{TT\}$	$\{TH, HT\}$	$\{HH\}$.
$p[D=r]$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E[D] = \tfrac{1}{4} \cdot 0 + 1 \cdot \tfrac{1}{2} + 2 \tfrac{1}{4} = 1.$$

Example : C # 8's in a hand of 5-cards.

Range - $\{0, 1, 2, 3, 4\}$

$$\cancel{P[C=0]\cdot 0} + P[C=1]\cdot 1 + P[C=2]\cdot 2 + P[C=3]\cdot 3 + P[C=4]\cdot 4$$

$$\frac{1}{\binom{52}{5}}\left[\binom{4}{1}\binom{48}{4} 1 + 2\cdot \underbrace{\binom{4}{2}\binom{48}{3}}_{P[C=r]} + 3\binom{4}{3}\binom{48}{2} + 4\binom{4}{4}\binom{48}{1}\right]$$

Example: Coin flipped 2 times

Q: $(\#\text{ heads})^2$

	Q
HH	4
HT	1
TH	1
TT	0

$E(Q)$

$$4\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 0\cdot\frac{1}{4}$$

$$= 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}.$$

Play costs \$1

Push button 1 : # 1 -- 10 each (x, y)
Push button 2 : # 1 ... 10. equally likely.

If the two #'s are same: win \$k.

How large should k be for you to be willing to play?

W = winnings

Range = $\{-1, k-1\}$

$$(-1) \cdot P[W = -1] + (k-1) P[W = k-1].$$

$P[(x,y) : x \neq y]$ $\quad \frac{90}{100}$ $\qquad$ $P[(x,y) : x = y]$ $\quad \frac{10}{100}$

$$(-1)\left(\frac{9}{10}\right) + (k-1)\frac{1}{10} = \frac{k}{10} - 1$$

$(k \geq 10$ or $k > 10)$?

Two spinners outcome $\{1,2,3\}$
Each (x,y) equally likely. $S = \{1,2,3\} \times \{1,2,3\}$

$P(x,y) = x \cdot y$. Range: $\{1,2,3,4,6,9\}$

$|S| = 9$.

What is $E[P]$?

$1 \cdot \underbrace{P[P=1]}_{\frac{1}{9}} + 2 \cdot \underbrace{P[P=2]}_{\frac{2}{9}} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{1}{9} + 6 \cdot \frac{2}{9} + 9 \cdot \frac{1}{9}$.

Team w/ 4 rookies
10 veterans.

Random set of 5 is chosen.

R = # rookies chosen.

Range = $\{0,1,2,3,4\}$.

$$E[R] = \cancel{0 \cdot P[R=0]} + 1 \cdot P[R=1] + 2\,P[R=2] + 3 \cdot P[R=3] + 4\,P[R=4]$$

$$= \qquad \frac{\cdot}{\binom{14}{5}}$$

$$= \frac{1}{\binom{14}{5}}\left[1 \cdot \binom{4}{1}\binom{10}{4} + 2 \cdot \binom{4}{2}\binom{10}{3} + 3 \cdot \binom{4}{3}\binom{10}{2} + 4 \cdot \binom{4}{4}\binom{10}{1}\right]$$

pairs in a 5-card hand.