

Recursive Definitions:

Defines something (sequence, function, set, procedure...) by defining larger instances in terms of smaller sets.

Induction: proof method.

(often used to prove facts about recursively defined objects).

Strings of '(' and ')'

$P \subseteq \{ (,) \}$ is the set of properly nested parens.

Basis: $() \in P$.

$$u = ()()$$

$$()()$$

Recursive Rules:

- ① if $u \in P$ then $(u) \in P$. $(())()$
- ② if $u + v \in P$ then $uv \in P$.

$(())()$ $()$ $(())$ $()()$ $(())()$ $(())()$

Theorem: If $x \in P$ then # left parens in x .
= # right parens in x .

Proof: Base case: left $[()]$ = right $[()]$ = 1

Inductive Step: Prove for $x \in P$ $\text{left}[x] = \text{right}[x]$

(1) Case 1 last rule to create x was Rule 1.

$x = (u)$ for smaller string u .

by the inductive hypothesis

$$\text{left}[u] = \text{right}[u].$$

$$\begin{aligned} \text{left}[x] &= \text{left}[(u)] = 1 + \text{left}[u] \\ &= 1 + \text{right}[u] \\ &= \text{right}[(u)] \\ &= \text{right}[x]. \end{aligned}$$

Case (2) ..

Example of a bijection.

P_6 = Set of all binary palindromes of length 6.

$$P_6 \subseteq \{0,1\}^6$$

$$101101 \in P_6$$

$$111101 \notin P_6$$

$$\text{Show: } |P_6| = |\{0,1\}^3| = 2^3$$

$$f: \{0,1\}^3 \rightarrow P_6$$

$$x \in \{0,1\}^3 \quad x = b_1 b_2 b_3$$

$$f(x) = b_1 b_2 b_3 b_3 b_2 b_1 \quad f(010)$$

1-1

If $x \neq x'$ $f(x) \neq f(x')$.

(Adding bits to $x + x'$ can not make them equal).

onto: $y \in P_6$ find $x \in \{0,1\}^3$ $f(x) = y$.

$$y = b_1 b_2 b_3 b_3 b_2 b_1 = b_1 b_2 b_3 b_3 b_2 b_1$$

$$\text{let } x = b_1 b_2 b_3 \quad f(x) = b_1 b_2 b_3 b_3 b_2 b_1 = y.$$

Strong Induction

Theorem: For any $n \geq 8$ can make n cents worth of stamps using only 3¢ + 5¢ stamps.

$P(n)$: can make n cents of stamps using only 3¢ + 5¢ stamps. P(7)?
P(6).

Pf: Base case: $n=8$: $P(8)$ is true (3¢ + 5¢).
 $n=9$: $P(9)$ is true (3¢, 3¢, 3¢).
 $n=10$: $P(10)$ is true (5¢, 5¢).

Inductive Step:

[If $(P(8) \wedge P(9) \wedge \dots \wedge P(k))$ for $k \geq 10$.
Then $P(k+1)$ is true.

$k=10$ $(P(8) \wedge P(9) \wedge P(10)) \rightarrow P(11)$.
 $k=11$ $(P(8) \wedge P(9) \wedge P(10) \wedge P(11)) \rightarrow P(12)$.
 $k=12$
⋮

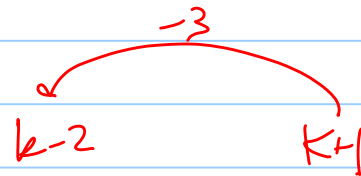
Weak
 $P(8) \rightarrow P(9)$
 $P(9) \rightarrow P(10)$
 $P(10) \rightarrow P(11)$

~~$(P(8) \wedge P(9)) \rightarrow P(10)$~~

[$P(j)$ is true for any $j=8, 9, \dots, k$]

Assume $k \geq 10$ and $P(j)$ is true for any $j = 8, 9, 10, \dots, k$.

Prove: $P(k+1)$.



If $k \geq 10$ then $k-2 \geq 8$.

Therefore $k-2$ is in the range $8, 9, \dots, k$.
By the inductive hypothesis, $P(k-2)$ is true.
Add a 3¢ stamp to get $(k-2+3)¢ = (k+1)¢$
worth of stamps.

Therefore $P(k+1)$ is true.

