Recursive Definition:

Defines something (sequence, function, set, procedure ...)
by defining larger instances in terms of smaller sets.

Induction : proof method.
(often used to prove facts about recursively defined objects).

Strings of '(' and ')'

$P \subseteq \{ (,) \}$ is the set of properly nested parens.

Basis: $() \in P$.

u = (())()

((())())

Recursive Rules:

(1) if $u \in P$ then $(u) \in P$.   ((())()

(2) if $u + v \in P$ then $uv \in P$.

(())|()

() (()) ()() ((())) (()())

Theorem: If $x \in P$ then # left parens in x = # right parens in x.

Proof: Base Case: left[()] = right[()] = 1

Inductive Step: Prove for $x \in P$ $left[x] = right[x]$

(1) Case 1 last rule to create x was Rule 1.

$x = (u)$ for smaller string u.

by the inductive hypothesis

$left[u] = right[u]$.

$$\begin{aligned} left[x] &= left[(u)] = 1 + left[u] \\ &= 1 + right[u] \\ &= right[(u)] \\ &= right[x]. \end{aligned}$$

Case (2) ..

Example of a bijection.

P_6 = Set of all binary palindromes of length 6.

$P_6 \subseteq \{0,1\}^6$

$101101 \in P_6$

$111101 \notin P_6$.

Show: $|P_6| = |\{0,1\}^3| = 2^3$

$f : \{0,1\}^3 \to P_6$.

$x \in \{0,1\}^3$ $\qquad x = b_1 b_2 b_3$

$f(x) = b_1 b_2 b_3 b_3 b_2 b_1$ $\qquad f(010)$

1-1

If $x \neq x'$ $f(x) \neq f(x')$.

(adding bits to x + x' can not make them equal).

onto:

$y \in P_6$ find $x \in \{0,1\}^3$ $f(x) = y$.

$y = b_1 b_2 b_3 b_4 b_5 b_6 = b_1 b_2 b_3 b_3 b_2 b_1$

let $x = b_1 b_2 b_3$ $\qquad f(x) = b_1 b_2 b_3 b_3 b_2 b_1 = y$.

Strong Induction

Theorem: For any $n \geq 8$ can make n cents worth of stamps using only 3¢ + 5¢ stamps.

$P(n)$: can make n cents of stamps using only 3¢ + 5¢ stamps.

$P(7)$? $P(6)$.

Pf: Base case: $n=8$: $P(8)$ is true (3¢ + 5¢).
$n=9$: $P(9)$ is true (3¢, 3¢, 3¢).
$n=10$: $P(10)$ is true (5¢, 5¢).

Inductive Step:
If $(P(8) \wedge P(9) \wedge \cdots \wedge P(k))$ for $k \geq 10$.
then $P(k+1)$ is true.

Strong

$k=10$ $(P(8) \wedge P(9) \wedge P(10)) \rightarrow P(11)$.
$k=11$ $(P(8) \wedge P(9) \wedge P(10) \wedge P(11)) \rightarrow P(12)$.
$k=12$
⋮

Weak
$P(8) \Rightarrow P(9)$
$P(9) \Rightarrow P(10)$
$P(10) \Rightarrow P(11)$
⋮

$(P(8) \wedge \cancel{P(9)}) \rightarrow P(10)$

[$P(j)$ is true for any $j = 8, 9, \ldots, k$]

Assume $k \ge 10$ and $P(j)$ is true for any
$j = 8, 9, 10, \ldots, k$.

$k-2 \xleftarrow{-3} k+1$

Prove: $P(k+1)$.

If $k \ge 10$ then $k-2 \ge 8$.

Therefore $k-2$ is in the range $8, 9, \ldots, k$.
By the inductive hypothesis, $P(k-2)$ is true.
Add a 3¢ stamp to get $(k-2+3)¢ = (k+1)¢$
worth of stamps.
Therefore $P(k+1)$ is true.

$P(8) \quad P(9) \quad P(10) \quad P(11) \quad P(12) \quad P(13)$