

Theorem: Every positive integer is greater than or equal to its reciprocal.

Translation: If x is an integer and $x > 0$ then $x \geq \frac{1}{x}$.

Implicit use of universal generalization.

Name a generic integer x and prove

$$(x > 0) \rightarrow (x \geq \frac{1}{x}).$$

Conclude $\forall x (x > 0) \rightarrow (x \geq \frac{1}{x})$ Domain: integers.

(Proof):) Let x be a positive integer. We will show that $x \geq \frac{1}{x}$.

- Since x is an integer and $x > 0$, then $x \geq 1$.

$$x \geq 1$$

Since x is positive, we can multiply both sides of the inequality by x :

$$\underline{x} \cdot \underline{x} \geq \underline{1} \cdot \underline{x} \quad \text{and therefore} \\ \underline{x^2} \geq \underline{x}.$$

Since $x^2 \geq x$ and $x \geq 1$, then $x^2 \geq 1$

Since x is positive, we can divide both sides of the inequality $x^2 \geq 1$ by x :

$$\frac{x^2}{x} \geq \frac{1}{x}$$

Therefore $x \geq \frac{1}{x}$. 

What can we assume in proofs? (depends on the audience).

* Basic algebra.

Some definitions:

Integer x is even iff $\underline{x = 2k}$ for some integer k .

Integer x is odd iff $\underline{x = 2k+1}$ for some integer k .

If x is an integer and x is not odd, then
 x is even.

If x is an integer and x is not even, then
 x is odd.

A rational number is a real number x
such that $x = \frac{a}{b}$ for some integers $a+b$
where $b \neq 0$.

If x is a real number and it is not
rational then x is irrational.

Most theorems we prove are universal statements:

"The square of every even integer is even."

⇒ For every integer x , if x is even then x^2 is even.

To prove a universal statement is false,

Every real number is less than its square.

If x is a real number then $x < x^2$.

Counter-examples: $x=0$ $x=\frac{1}{2}$.

If the product of two integers is a multiple of four then both numbers must be even.

If x and y are integers
and $xy = 4k$ for some integer.
then $x + y$ are even.

Counter-example $x=4$ $y=1$.

Proving Conditional Statements.

Note Title

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Many theorems have the form : $h \rightarrow c$
or $\forall x \underline{H(x)} \rightarrow \underline{C(x)}$.

\Rightarrow The cube of every negative number is negative.

For all x , $(x < 0) \rightarrow (x^3 < 0)$.

Domain: Set of real numbers.

\Rightarrow If x^2 is an odd integer then x is an odd integer.

For all x , $(x^2 \text{ is odd}) \rightarrow (x \text{ is odd})$

Domain: Set of integers.

Ways to prove $\forall x H(x) \rightarrow C(x)$.

Direct Proof:

- Name a generic element of the domain that satisfies $H(x)$
- Prove that $C(x)$ is true.

Proof by contrapositive.

- Name a generic element of the domain such that $C(x)$ is false
- Prove that $H(x)$ is also false.
- Must state at the beginning what you are assuming & what you are proving.

Proofs by contrapositives work because

$$\Rightarrow \forall x H(x) \rightarrow C(x) \equiv \forall x \neg C(x) \rightarrow \neg H(x).$$

If you have more than one hypothesis, only need to show that one is false:

$$\begin{aligned} \forall x & \left(\underbrace{(H_1(x) \wedge H_2(x))}_{\text{one}} \rightarrow C(x) \right) \\ & \equiv \forall x \left(\underbrace{(H_1(x) \wedge \neg C(x))}_{\text{one}} \rightarrow \underbrace{\neg H_2(x)}_{\text{one}} \right). \end{aligned}$$

Direct proof example:

Theorem: The cube of every even integer is even.

Proof: Let x be an even integer. We will show that x^3 is also even.

\Rightarrow Since x is even, $x = 2k$ for some integer k .
Cube both sides of the equation to get:

$$x^3 = (2k)^3 = 2^3 k^3 = 8k^3 = 2(4k^3).$$

?

Since k is an integer, $4k^3$ is also an integer, so x^3 can be expressed as two times an integer and therefore x^3 is even. \square

Theorem : If $\underline{x < 2}$ then $\overbrace{x^2 - 9x + 14 > 0}$.
 $\overbrace{(x-7)(x-2) > 0}$

$$(x-2) < 0$$

Want: $(x-7) < 0$.

If $x < 2$ then $x < 7$

$$(x-7) < 0.$$

Proof: Assume that $\underline{x < 2}$ and prove that $\overbrace{x^2 - 9x + 14 > 0}$.

- Since $x < 2$ then $x-2 < 0$.
- Since $x < 2$ then $x < 7$ and $x-7 < 0$.

The product of two negative numbers is positive, so

$$\underline{(x-2)(x-7) > 0} \text{. Since } \underline{(x-2)(x-7)} = \underline{x^2 - 9x + 14},$$
$$\underline{x^2 - 9x + 14 > 0}. \quad \square$$

Theorem: The product of two rational numbers is also rational.

Proof: Let x and y be rational numbers. We will show that xy is also rational.

Since x and y are rational,

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d} \quad \text{for some integers } a, b, c, d \text{ where } b \neq 0 \text{ and } d \neq 0.$$

Multiply x and y to get:

$$xy = \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}.$$

ac and bd are both integers. Also $bd \neq 0$ because $b \neq 0$ and $d \neq 0$.

Therefore xy is the ratio of two integers with a non-zero denominator which means that xy is rational. \square

Proof by Contrapositive example:

Theorem : If n is an integer and $3n+7$ is odd
then n is even.

Proof : Let n be an odd integer. We will show
that $3n+7$ is even.

Since n is odd, $n = 2k+1$ for some
integer k .

Plug the expression $2k+1$ into $3n+7$

$$\begin{aligned}3n+7 &= 3(2k+1)+7 = 6k+3+7 \\&= 6k+10 = 2(3k+5)\end{aligned}$$

Since k is an integer, then $3k+5$ is also
an integer. Therefore $3n+7$ can be expressed
as 2 times an integer and therefore $3n+7$
is even. \square .

Theorem : If x is a real number such that $3x$ is irrational then x is irrational.

Before we start

Every real number is rational or irrational.
Every rational number is real.

A number is irrational if and only if it is real and not rational.

Proof : Assume that x is a real number and x is not irrational.

We will show that $3x$ is rational and therefore not irrational.

Since x is real and not irrational then x is rational.

Therefore $x = \frac{a}{b}$ where a and b are integers and $b \neq 0$.

$$3x = 3 \cdot \frac{a}{b} = \frac{3a}{b}$$

Since a is an integer $3a$ is also an integer. Therefore $3x$ can be expressed as the ratio of two integers with a non-zero denominator, which means that $3x$ is rational. \square