

Functions

Note Title

10/10/2014

Continuous functions:

$$f(x) = x^2 - 3$$

$$f(x) = \sin^2(x)$$

Discrete math: functions whose inputs + outputs are from finite or countably infinite sets.

Example: map a set of computational tasks to computers in a distributed network.

The input/output relationship of a digital circuit. (or computer program)

First part of definition: input & output sets.

$$f: A \rightarrow B$$

A is the domain } A + B are sets.
B is the target }
↳ co-domain.

$$f: \{0,1\}^5 \rightarrow \{0,1\}^4$$

$$f(01101) = 1001$$

input: binary strings length 5

output: binary strings of length 4.

f must be well-defined for each element in the domain.
 ↳ one output in range for each input.

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\rightarrow f(x) = 1/x$$

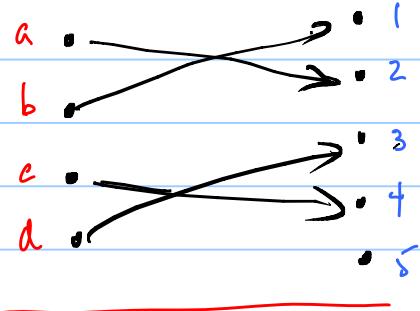
$$f(x) = \pm \sqrt{x^2 + 1}$$

Not defined for $x=0$.

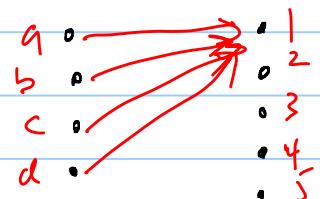
Two different outputs
for each input.

Arrow diagram.

$$X = \{a, b, c, d\} \quad g: X \rightarrow Y$$
$$Y = \{1, 2, 3, 4, 5\}$$



$$\begin{aligned}f(a) &= 2 \\f(b) &= 1 \\f(c) &= 4 \\f(d) &= 3\end{aligned}$$



Range of a function $f \subseteq \text{Target}$. $f: X \rightarrow Y$

$$\text{Range of } f = \{y \in Y : \exists x \in X \quad f(x) = y\}$$

For example above: $\{1, 2, 3, 4\}$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\underline{f(x) = x^2}$$

range: non-negative reals.

$$g: \mathbb{Z} \rightarrow \mathbb{Z}$$

$$\underline{f(x) = 3x}$$

range: multiples of 3

$$f: \overline{\{0, 1\}^2} \rightarrow \overline{\{0, 1\}^3}$$

$f(x)$: copy the first bit and append to the end.

$$\overbrace{\hspace{1cm}}^{\text{L}} \{00, 01, 10, 11\}$$

$$f(00) = \overset{\downarrow}{0}00$$

$$\text{Range: } \{000, 010, 101, 111\}$$

$$f(01) = 010$$

$$f(10) = 101$$

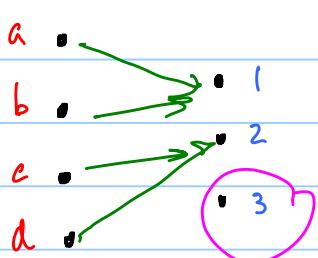
Range? Target

$$f(11) = 111$$

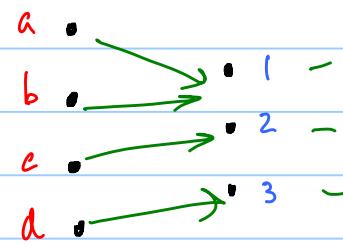
NO.

A function $f: X \rightarrow Y$ is onto if the range of $f = Y$.
 $\forall y \in Y \exists x \in X f(x) = y$.

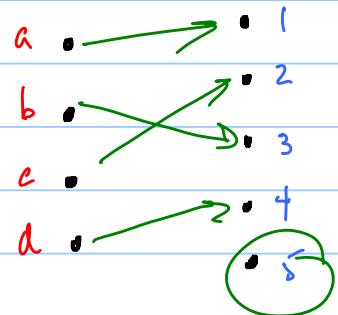
$$X = \{a, b, c, d\} \quad Y = \{1, 2, 3\}$$



Not onto



ONTO



Note : if $|Y| > |X|$ then f can not be onto.
 if f is onto then $|Y| \leq |X|$.

Onto Examples :

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2$$

Not onto.

No x s.t. $f(x) < 0$

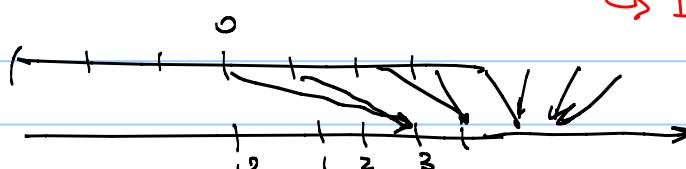
$$f: \mathbb{Z} \rightarrow \mathbb{Z} \quad f(x) = 2x + 3$$

Not onto.

No $\exists x \in \mathbb{Z}$ $f(x)$ is even

$$f: \mathbb{N} \rightarrow \mathbb{Z} \quad f(x) = x/2 + 3$$

Integer division - drop the remainder.

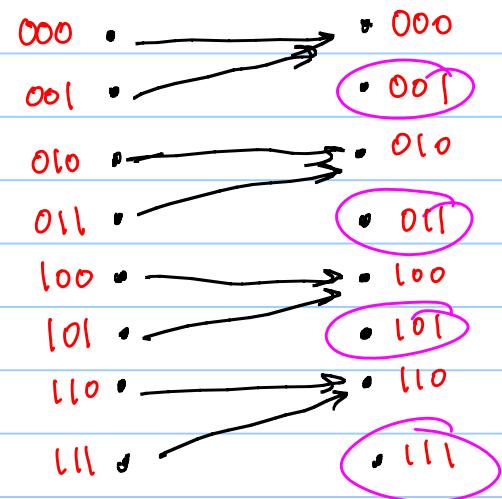


- $f: \overline{\{0,1\}^3} \rightarrow \overline{\{0,1\}^3}$

$$f(011) = 010.$$

replace the last bit
with 0.

NOT onto.

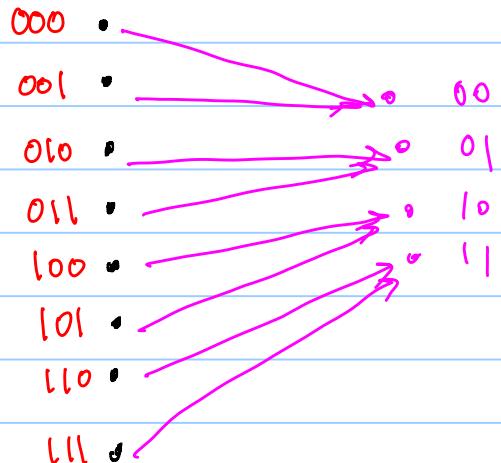


- $f: \overline{\{0,1\}^3} \rightarrow \overline{\{0,1\}^2}$

drop the last bit

$$f(011) = 01.$$

ONTO.

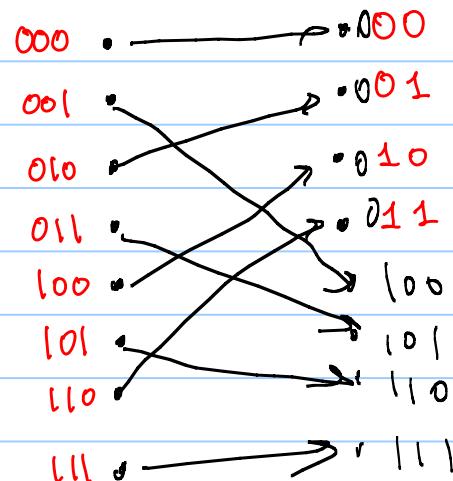


- $f: \overline{\{0,1\}^3} \rightarrow \overline{\{0,1\}^3}$

remove the last bit and add
it to the beginning of the string.
(Cyclic shift right).

$$f(110) = 011$$

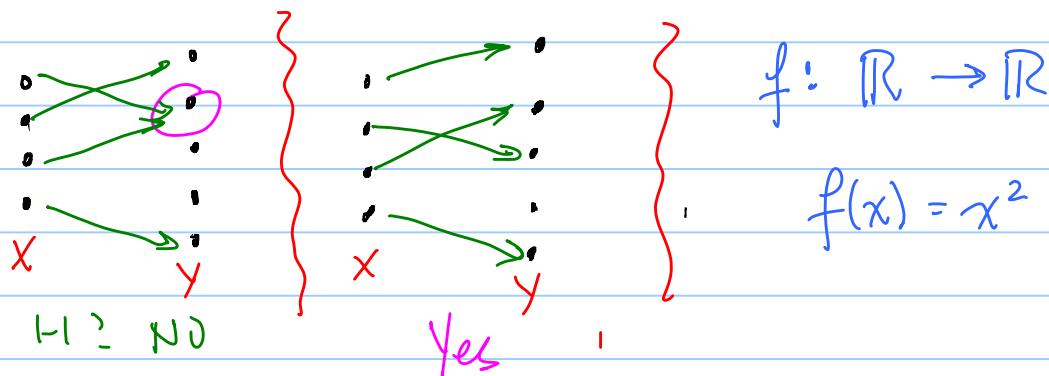
ONTO!



$f: X \rightarrow Y$

A function f is 1-1 (one-to-one) if no two elements in the domain map on to same element in the range.

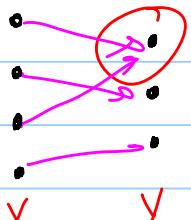
$$x, x' \in X \quad x \neq x' \Rightarrow f(x) \neq f(x').$$



$$g: X \rightarrow Y$$

If g is one-to-one then $|Y| \geq |X|$.

If $|Y| < |X|$ then g is not one-to-one.



Recap: $f: \text{Set of employees} \rightarrow \text{Set of offices}$ $f: \text{Office assignm.}$
e.g. $f(\text{Sam}) = \text{Rm 112}$.

If f is onto then no office is unoccupied.

If f is 1-1 then everyone gets their own office.

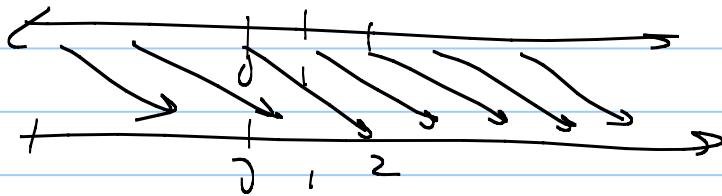
$$g: \mathbb{R} \rightarrow \mathbb{R}$$

$$\underline{g(x) = x^2}$$

$$f(-3) = f(3)$$

$$g: \mathbb{Z} \rightarrow \mathbb{Z}$$

$$\underline{g(x) = x+2}$$



$$\underline{g(x) = 2x}$$

1-1 w/ onto.

What about

$$g: \mathbb{N} \rightarrow \mathbb{N}$$

$$g(x) = x+2.$$

1-1

w/ onto.

$$g(x) = x/2 \quad (\text{integer division, throw away remainder})$$

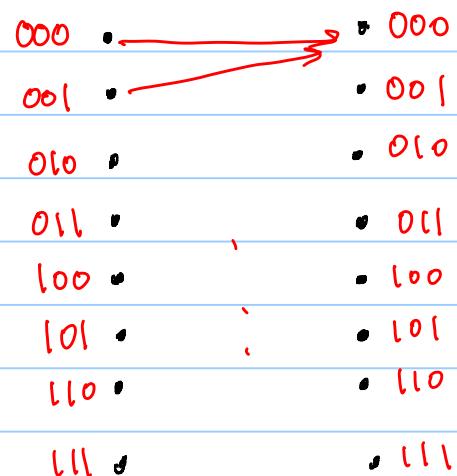
$$g(0) = g(1) = 0.$$

- $f : \{0,1\}^3 \rightarrow \{0,1\}^3$

$$f(011) = 010.$$

replace the last bit
with 0.

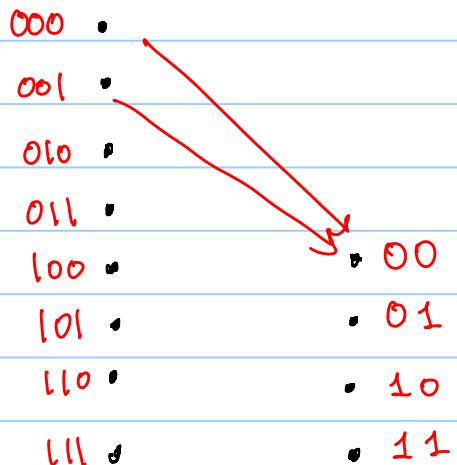
Not (-1) $f(000) = f(001)$.



- $f : \{0,1\}^3 \rightarrow \{0,1\}^2$

drop the last bit

$$f(011) = 01.$$

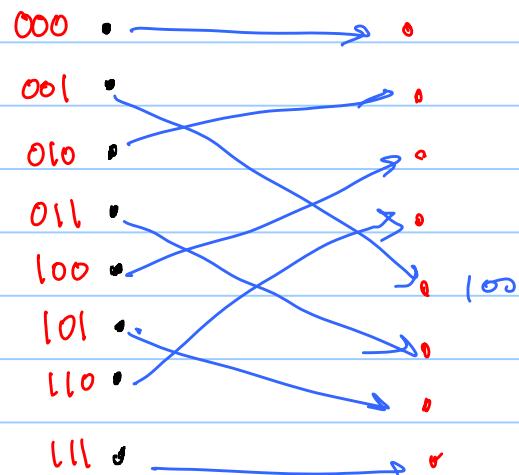


- $f : \{0,1\}^3 \rightarrow \{0,1\}^3$

remove the last bit and add
it to the beginning of the string.
(Cyclic shift right).

$$f(110) = 011$$

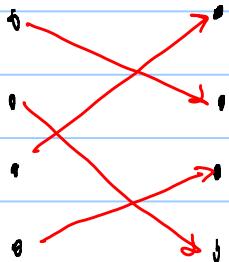
|-| and onto.



If f is 1-1 and onto it is a bijection.
(one-to-one correspondence).

If $f: X \rightarrow Y$ is one-to-one $\Rightarrow |Y| \geq |X|$.
onto $\Rightarrow |X| \geq |Y| \quad \left. \begin{array}{l} \\ \end{array} \right\} |X|=|Y|$

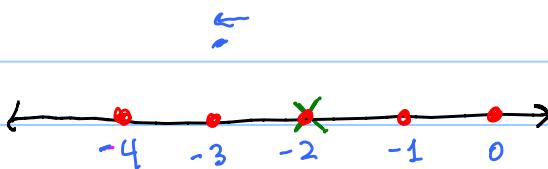
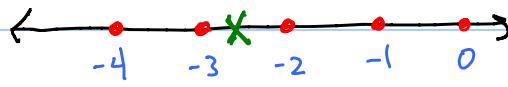
If $|X| \neq |Y|$ then f is not a bijection.



A bijection pairs up elements
in the domain and elements
in the target.

Floor function: $\text{floor}: \mathbb{R} \rightarrow \mathbb{Z}$.

$$\lfloor -2.7 \rfloor = -3$$



$$\lfloor -2 \rfloor = -2$$

$$\text{floor}(x) = \lfloor x \rfloor =$$

largest integer that is less than or equal to x .

$$\lfloor 3.1 \rfloor = 3$$

$$\lfloor -2 \rfloor = -2$$

$$\lfloor -3.1 \rfloor = -4$$

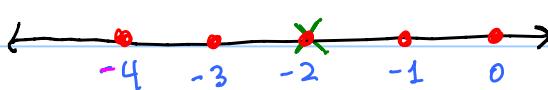
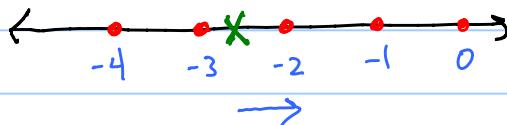
$$\lfloor 4.9 \rfloor = 4$$

$$\lfloor 0 \rfloor = 0$$

$$\lfloor -5.9 \rfloor = -6$$

Ceiling function: $\text{ceil} : \mathbb{R} \rightarrow \mathbb{Z}$.

$$\lceil -2.7 \rceil = -2$$



$$\lceil -2 \rceil = -2$$

$$\text{ceil}(x) = \lceil x \rceil =$$

smallest integer that is greater than or equal to x .

$$\lceil 3.1 \rceil = 4$$

$$\lceil -2 \rceil = -2$$

$$\lceil -3.1 \rceil = -3$$

$$\lceil 4.9 \rceil = 5$$

$$\lceil 0 \rceil = 0$$

$$\lceil -5.9 \rceil = -5$$

Functions can have more than one input:

$$f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$$

$$f(x, y) = x + y$$

$$\underline{f(x, y)} = \max \{\underline{x}, \underline{y}\}.$$

$$\underline{f(x, y)} = \underline{2^x + 2^y}$$

or $g: \underline{\mathbb{Z}} \times \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{Z}} \times \underline{\mathbb{Z}}$

$$g(x, y) = (\max \{\underline{x}, \underline{y}\}, \min \{\underline{x}, \underline{y}\}).$$

$$g(\underline{x}, \underline{y}) \stackrel{?}{=} (3, 6) \quad \text{Not onto}$$

$$f(3, 6) = f(6, 3) \quad \text{Nor 1-1.}$$

Inverse & Composition of functions.

Note Title

2/4/2015

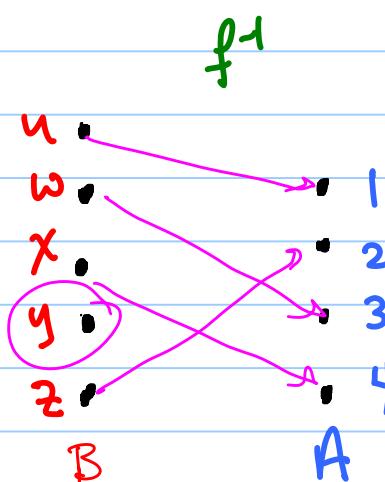
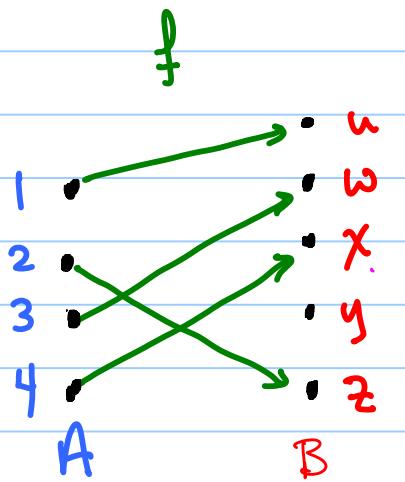
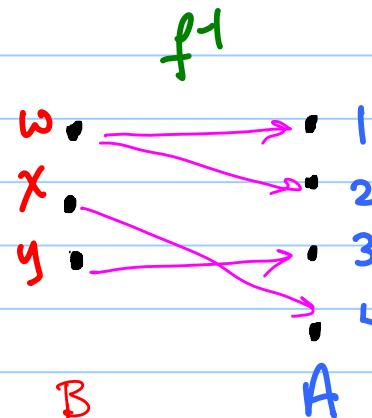
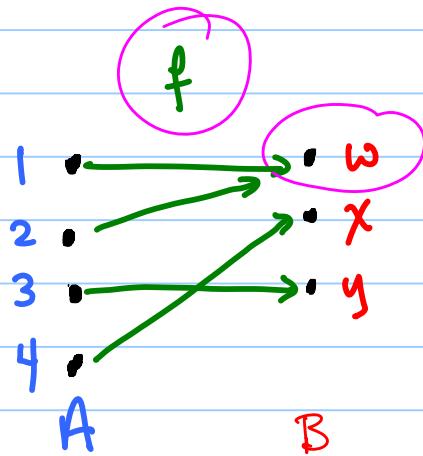
let $f: A \rightarrow B$

Define the inverse of f (denoted f^{-1})

$$f(a) = b \quad \text{iff} \quad f^{-1}(b) = a.$$

for all $a \in A \& b \in B$.

f^{-1} is not always a well defined function:





Theorem A function f has a well-defined inverse if and only if it is a bijection.

Examples: $f: \mathbb{Z} \rightarrow \mathbb{Z}$

$$f(x) = \underline{5x+3} \quad \text{Not onto}$$

$$\underline{\underline{f(x) = x-2}} \quad \begin{aligned} y &= x-2 \\ \underline{y+2} &= x \end{aligned}$$

$$\underline{\underline{f(x) = |x|}} \quad \text{neither.}$$

$$f(x) = \underline{\underline{\left\lfloor \frac{x}{2} \right\rfloor}} \quad \text{Not } 1-1.$$

$$f(y) = y+2 \quad f(x) = x+2$$

One way to show that a function is a bijection is to show its inverse.

Let $A = \{1, 2, 3\}$.

$f: \{0,1\}^3 \rightarrow P(A)$

for $i=1, 2, 3$

$i \in f(x)$

if and only if the i^{th} bit of x is 1.

$$f(101) = \{1, 3\}$$

$$f(010) = \{2\}$$

$$f(000) = \emptyset \text{ on } \{\}$$

<u>1</u>	<u>2</u>	<u>3</u>
0	0	0

Is f a bijection?

$f^{-1}: P(A) \rightarrow \{0,1\}^3$

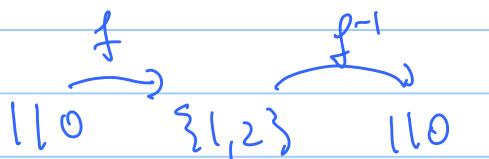
Let $X \subseteq A$

$f^{-1}(X) = b_1 b_2 b_3$

for $i=1, 2, 3$

$b_i = 0$ if $i \notin X$
 $= 1$ if $i \in X$.

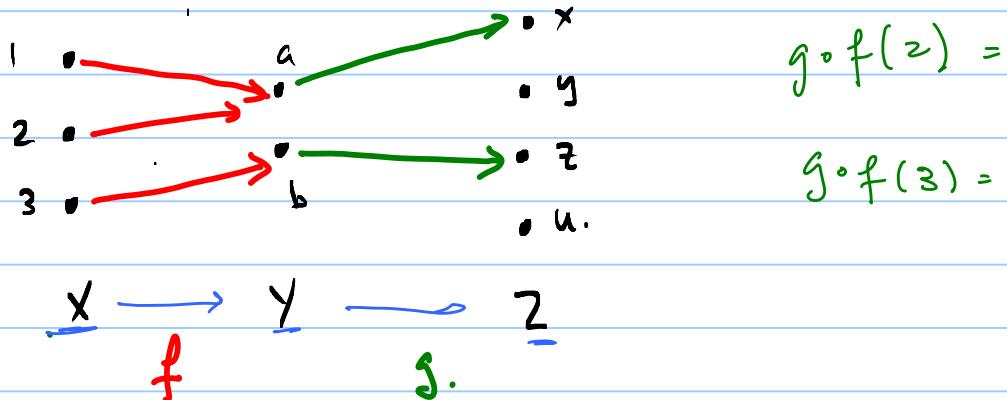
$$f^{-1}(\{1, 2, 3\}) = 111$$



$$f^{-1}(\{1\}) = 100$$

For any $X \subseteq A$ $f^{-1}(X)$ corresponds to exactly one string in $\{0,1\}^3$.

Composition of functions: applying one function
to their another.



$$f: X \rightarrow Y$$

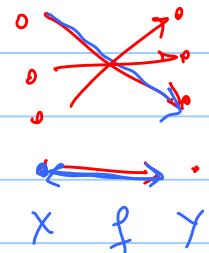
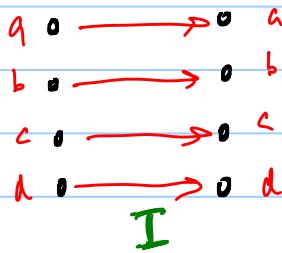
$$g: Y \rightarrow Z$$

$$g \circ f: X \rightarrow Z \quad x \in X \quad g(f(x))$$

$$\text{for } x \in X \quad g \circ f(x) = g(\underline{f(x)})$$

Identity function: $I: A \xrightarrow{\sim} A$ Domain = Target.

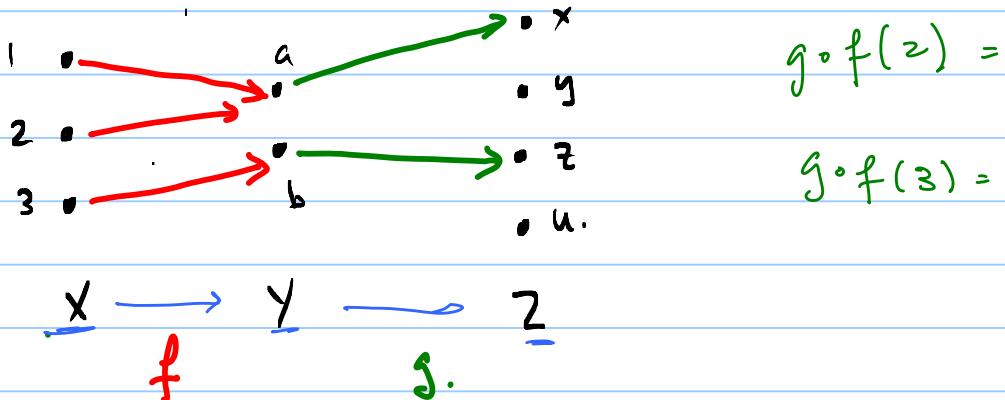
For all $a \in A$ $I(a) = a$.



$f: X \rightarrow Y$ bijection. $f^{-1}: Y \rightarrow X$. id

$$\text{id.} \rightarrow f^{-1} \circ f: X \rightarrow X \quad | \quad f \circ f^{-1}: Y \rightarrow Y$$

Composition of functions: applying one function
to their another.



$$f: X \rightarrow Y$$

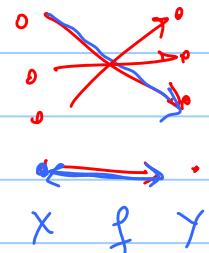
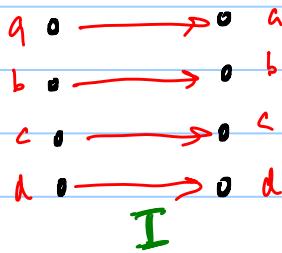
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