Binary Relations.

Note Title 2/13/2015

Set A & B

Binary Relation on/between A & B is a subset of $A \times B$.

$$R \subseteq A \times B$$

$$(a,b) \in R \iff aRb.$$

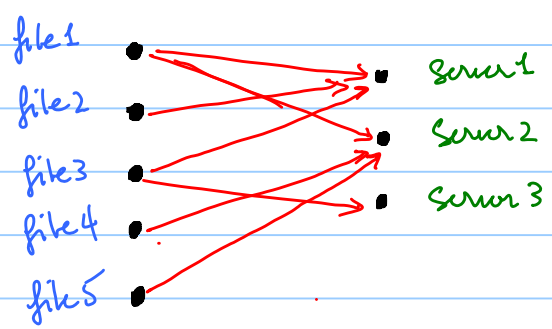
Example: Files in a large database stored in a distributed network.

F = set of all files.
S = set of all servers in the network.

Relation R to express whether a particular file f stored on a particular server s.

fRs $\quad$ f stored on s.

There can be more than one copy of a particular file stored on different servers (for fault tolerance).



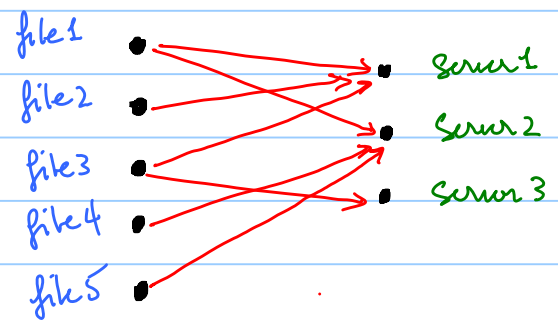
$R = \{ (f1, s1), (f1, s2), (f2, s1), (f3, s1), (f3, s3), (f4, s2), (f5, s2) \}$.

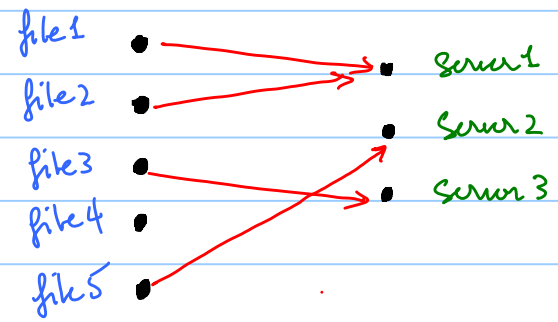
$F = \{ f1, f2, f3, f4, f5 \}$
$S = \{ s1, s2, s3 \}$

A relation R between sets F + S is mathematically the same as a predicate w/ two input variables:
F domain of x + S domain of y:

For all $f \in F \wedge s \in S$, $R(f,s) = \text{true}$ iff $(f,s) \in R$.

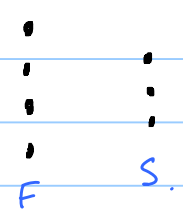
A Relation is a generalization of a function.
Every function $f: F \to S$ is a relation between F + S
Some relations between F + S are not well defined functions.



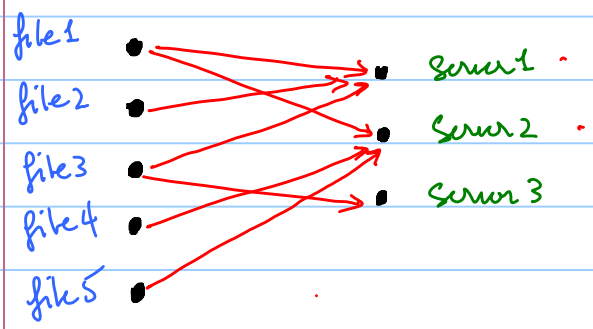


These are both relations but not functions.

A Relation can be empty:



A relation on finite sets can be represented by a matrix:



$$\begin{array}{c|ccc} & \text{serv 1} & \text{serv 2} & \text{serv 3} \\ \hline \text{file 1} & 1 & 1 & 0 \\ \text{file 2} & 1 & 0 & 0 \\ \text{file 3} & 1 & 0 & 1 \\ \text{file 4} & 0 & 1 & 0 \\ \text{file 5} & 0 & 1 & 0 \end{array}$$

$R \subseteq F \times S$: rows $\rightarrow$ elements in F

columns $\rightarrow$ elements in S.

Relations can be between infinite sets:

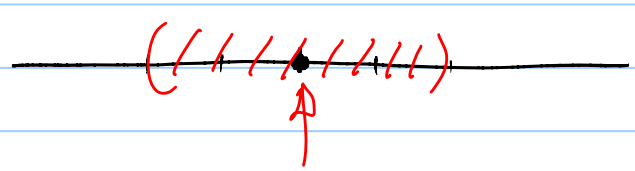
Relation C between $\mathbb{Z}$ and $\mathbb{R}$.

$x \, C \, y$ iff $|x-y| < 2$. Subset of $\mathbb{Z} \times \mathbb{R}$.

$\rightarrow 3 \not C 5.2$ $\quad |3-5.2| = |-2.2| = 2.2 \not< 2.$

$3 \, C \, 1.1$

$-4 \, C \, -5.5$ $\quad |-4 - -5.5| = |1.5| < 2.$



Relation R between a set A & itself is a subset of $A \times A$.

"R is a binary relation on the set A".

A is the domain of R.

Examples: Relation P on $\mathbb{N}$.

aPb iff there is an $n \in \mathbb{N}$ such that $b = a^n$

"b is a power of a".

$3 P 9$ yes. $3^2 = 9$

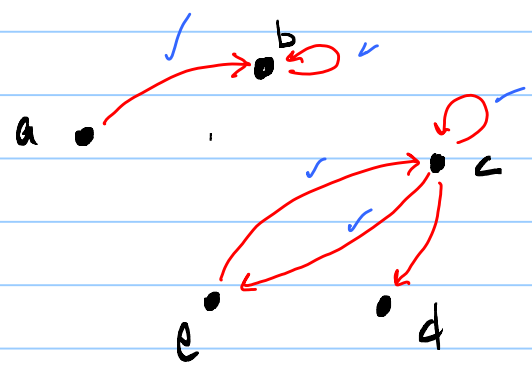
$9 \not P 3$ $9^{\square} = 3$

$2 P 16$ yes. $2^4 = 16$

$2 \not P 20$.

$4 P 4$

$A = \{a, b, c, d, e\}$



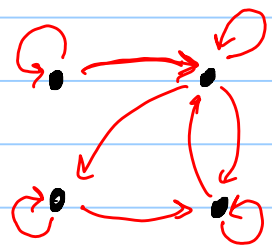
R.

$R = \{(a,b), (b,b), (e,c), (c,e), (c,c), (c,d)\}$

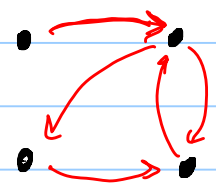
Properties of Relations: Relation R on A

Reflexive: $\forall a \in A$, $(a,a) \in R$.

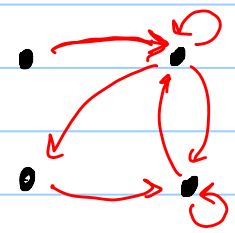
Anti-Reflexive: $\forall a \in A$, $(a,a) \notin R$.



reflexive



anti-reflexive.



neither.

S = a set of people.

Relation: M $\quad sMt$ if s & t have the same birthday.

Reflexive.

E $\quad sEt$ if s earns more money than t.

anti-reflexive.

M $\quad sMt$ if s sent an email to t yesterday.

neither.

Relation R on $\mathbb{R}$. $\quad xRy$ if $|x| = y$.

$2 R 2$? $\quad$ neither.

$-2 \not R -2$?

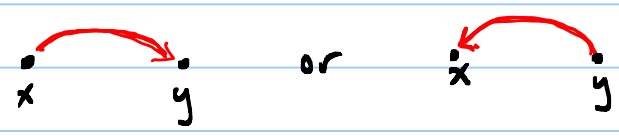
Symmetric / Anti-Symmetric. Relation R on A.

Symmetric: for every $x, y \in A$ $\quad xRy \rightarrow yRx$

trivially true if $x=y$.

For $x \neq y$:

Never have:



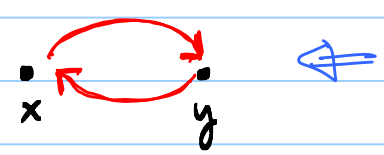
xRy : x is a sibling of y.
Then y is a sibling of x.

Anti-Symmetric: For every $x, y \in A$

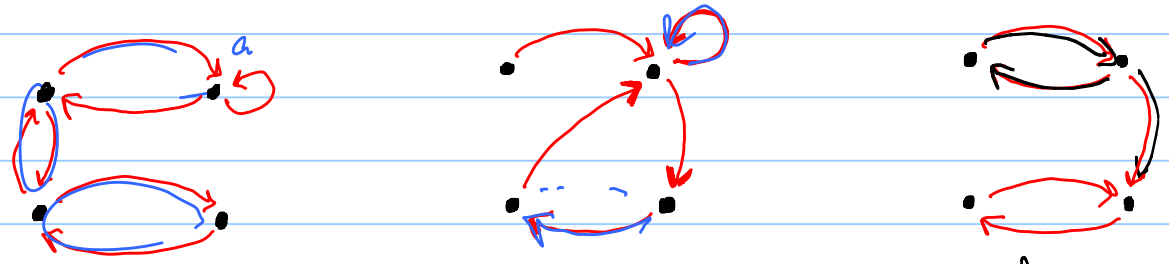
xRy and $yRx \rightarrow x=y$.

equivalent to xRy and $x \neq y \rightarrow y \not R x$

Can never have:



(if aRa and aRa then $a=a$)



Sym. $\qquad$ Anti-Sym. $\qquad$ Neither.

xMy $\quad$ x is a parent of y.
Anti-Symmetric

Anti-Symmetric!

P on $\mathbb{Z}$ $\quad xPy$ iff $\exists\, n \in \mathbb{N}$ $\quad x^n = y$.

$2P2$? $\quad 2^{\square} \overset{?}{=} 2$

$x^1 = x$

Reflexive.

$2P4$? yes $2^2 = 4$

$4P2$? no.

Not symmetric.

xPy $\wedge$ yPx.

$x^n = y$ and $y^m = x$

$\Rightarrow$ then $x = y$.

$(y^m)^n = y$.

$y^{mn} = y$.

$mn = 1$

$\Rightarrow x = y$.

L on $\mathbb{R}$ $\quad xLy$ iff $x < y$.

$x \not< x$ $\quad$ anti-reflexive.

if $x < y$ and $y < x$ then $x = y$.

anti-symmetric.

E on $\mathbb{R}$ $\quad xEy$ if $x \leq y$.

Reflexive.

Anti-symmetric

if $x \leq y$ and $y \leq x$ then $x = y$. yes.

S on $\mathbb{R}$ $\quad xSy$ iff $x^2 < y$.

Neither reflexive or anti-reflexive

$\frac{1}{3}\ S\ \frac{1}{4}$
$\frac{1}{4}\ S\ \frac{1}{3}$ $\Big\}$ not anti-symm.

$2\ S\ 5$
$5\ \not S\ 2$ $\Big\}$ not symmetric.

$\frac{1}{2}\ \overset{?}{S}\ \frac{1}{2} \Leftrightarrow \left(\frac{1}{2}\right)^2 < \frac{1}{2}$.

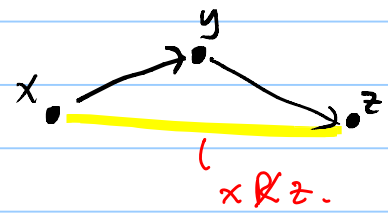
yes.

$3\ \not S\ 3$

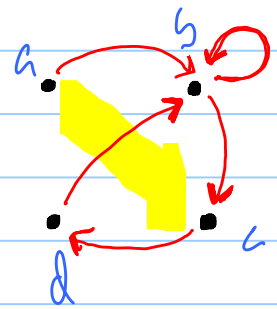
A relation R on set S is transitive if

$$xRy \text{ and } yRz \Rightarrow xRz.$$

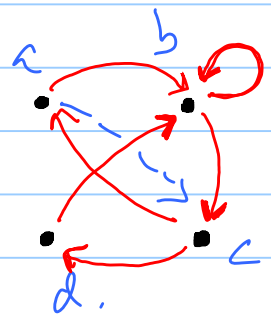
Never have:



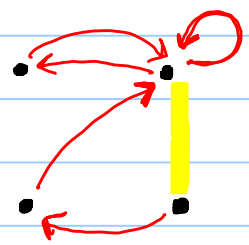
not trans.



not trans



not-trans



P on $\mathbb{Z}$ $\quad xPy$ iff $\exists\, n \in \mathbb{N}$ $x^n = y$. transitive

xPy $\quad yPz$

$x^n = y$ $\quad y^m = z$.

$(x^n)^m = z \Rightarrow xPz.$

$x^{nm} = z.$

M $\quad sMt$ if s & t have the same birthday.

true!

E $\quad sEt$ if s earns more money than t.

transitive.

P on $\mathbb{Z}$ $\quad$ aPb iff a & b are relatively prime

$\Rightarrow$ M $\quad$ sMt if s sent an email to t yesterday.

Not necessarily transitive.

D on $\mathbb{Z}^+$ $\quad$ aDb if a evenly divides b.

$b = ak \quad k \in \mathbb{Z}^+$

b is an integer multible of a.

$2R20$? $\quad$ $\underline{20}R\,\underline{60}$

$\Rightarrow 2R60.$

aDb and bDc then aDc?