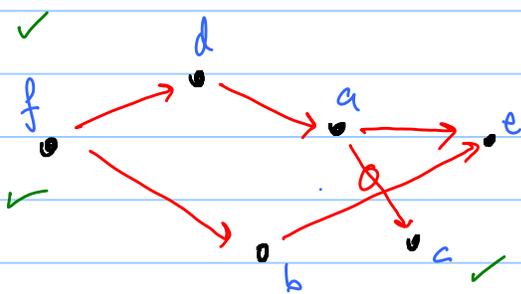


Directed Acyclic Graph: directed graph with no positive length cycles.

If  $G$  is a DAG then  $G^+$  is a strict order.

A topological sort of a DAG is an ordering of the vertices such that for every edge  $(u, v)$ ,  $u$  comes before  $v$  in the ordering.



f, d, a, c, b, e

OK

f, c, d, b, a, e

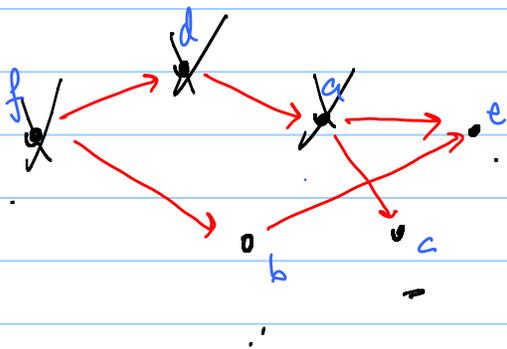
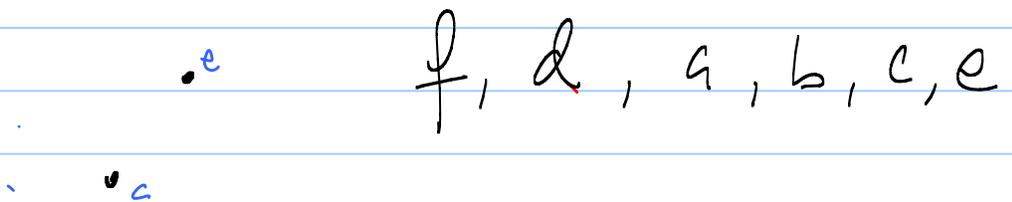
NO

f, b, d, a, e, c

A topological sort for a DAG is not necessarily unique.

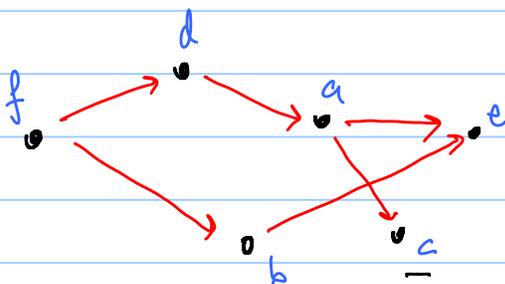
To find a topological sort, keep removing vertices that are minimal (i.e. in-degree = 0).

Order of removal is a topological sort.



f, b, d, a, e, c

f, d, a, b, e, c

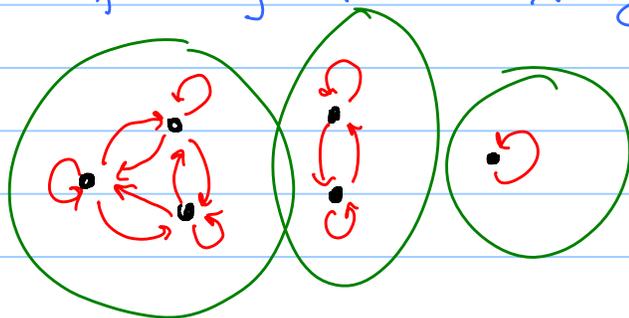


# Equivalence Relations.

A Relation  $R$  on a set  $A$  is an equivalence relation if it is:

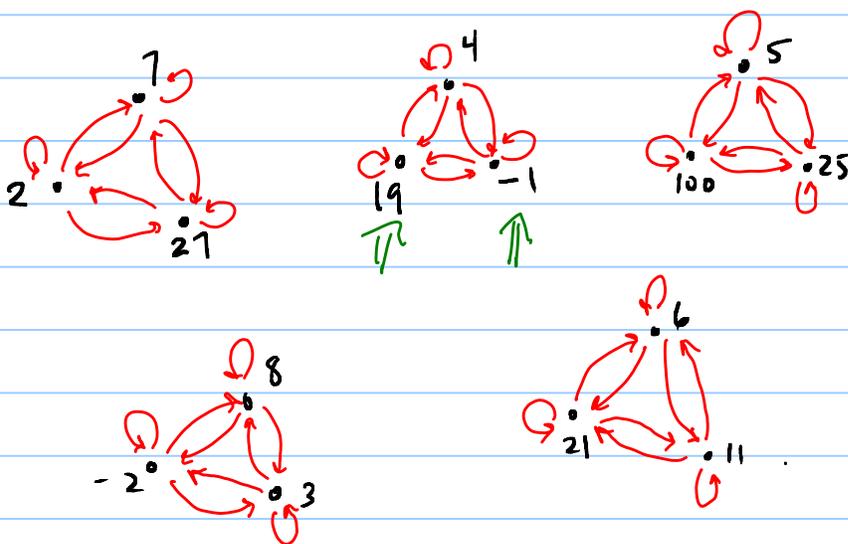
- reflexive
- Symmetric
- transitive!

if  $xRy$  then  $x \sim y$ .



Example: domain  $\mathbb{Z}$

$x \sim y$  iff  $x - y$  is a multiple of 5.



$x \sim y$  iff  $(x-y) = 5n$  for integer  $n$ .

Reflexive:  $x \sim x$   $(x-x) = 5 \cdot n$  for  $n \in \mathbb{Z}$   
 $n=0$  ✓

Symmetric:  $(x-y)$  is a mult of 5  $(x-y) = 5n$  ✓  
is  $(y-x)$  is a mult of 5.  $(y-x) = 5(-n)$ .

Transitive:

if  $(x \sim y$  and  $y \sim z)$  then  $x \sim z$  ?

$$\begin{array}{r} x-y = 5n \\ + \quad y-z = 5m \\ \hline x-z = 5n + 5m = 5(n+m) \end{array}$$

Group of people  $x \sim y$  if  $x$  and  $y$  have the same birth day.

Reflexive:  $x \sim x$ .

Symmetric:  $x \sim y \rightarrow y \sim x$

Transitive:  $x \sim y \wedge y \sim z \Rightarrow x \sim z$

Equivalence Class of  $x$ :  $[x] = \{y \mid x \sim y\}$ .

The set of distinct equivalence classes forms a partition of the underlying set.

How many equivalence classes for the birthday equivalence relation?

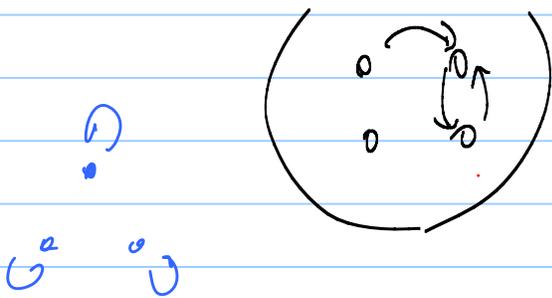
... as many as 366.

What are the equivalence classes of  
 $x \sim y$  iff  $(x-y)$  is a multiple of 5.

- [0] 100
- [1] 51
- [2] 17
- [3]
- [4]

Symmetric: Never see   
 $\forall x \forall y \quad x R y \rightarrow y R x$

Anti-symmetric: Never see   
 $\forall x \forall y \quad (x R y \wedge y R x) \rightarrow (y = x)$



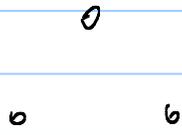
$$\leq \quad (x \leq y \text{ and } y \leq x) \rightarrow y = x$$

$$< \quad \underline{(x < y \text{ and } y < x) \rightarrow y = x}$$

Transitive  $\forall x \forall y \forall z \quad [(x R y \wedge y R z) \rightarrow x R z]$

Suppose  $\exists x \ y \ z$

$x R y$  and  $y R z$  and  $x R z$  Not transitive.



anti-reflex.

Sym, anti-Sym  
trans.

Non reflexive.



$$xRy \quad \text{if} \quad \frac{x-y = \text{rational}}{= \frac{a}{b} \quad \begin{matrix} a, b \in \mathbb{Z} \\ b \neq 0. \end{matrix}}$$

1. Reflexive  $xRx$

$(x-x)$  is rational. ✓  
0 is rational.

2. Sym:  $xRy \rightarrow yRx$ . ✓

if  $(x-y)$  is rational then  $(y-x)$  is rational.  
 $= \frac{a}{b}$   $= \frac{-a}{b}$

3.  $(xRy \wedge yRz) \rightarrow (xRz)$ .

$(x-y) = r_1$

+  $(y-z) = r_2$

$x-z$  rational?

$= r_1 + r_2$

transitive.

$x-z = r_1 + r_2$

$$x R y \text{ iff } \underline{x = 2y.}$$

6R3 ?  
~~3R6 ?~~

1 Reflexive:

$$x R x \\ \downarrow \\ x = 2x$$

0R0 not anti-refl.  
 3R3 not reflex.

$$x = 0.$$

2.

$x R y$  and  $y R x$  → then  $x=0$  and  $y=0$ .

$$x = 2y \quad y = 2x \\ x = 2(2x) \\ x = 4x \quad x = 0.$$

Anti-Sym ✓  
 $(x R y \wedge y R x) \rightarrow x = y.$

Transitivity

$$(x R y \wedge y R z) \rightarrow (x R z) \\ (4 R 2 \wedge 2 R 1) \rightarrow 4 R 1$$

$x = 4z$

Not transitive.

$$f: D \rightarrow T \\ \uparrow$$

$f \circ g$

$f(g(x))$

$g: X \rightarrow Y$   
 $f: Y \rightarrow Z$

Domain: X  
 Target: Z.