

Boolean Functions

$$f: \{0,1\}^n \rightarrow \{0,1\}$$

Maps one or more boolean variables to 0/1.

Can specify a boolean function by an output table

<u>x</u>	<u>y</u>	<u>$f(x,y)$</u>
0	0	0 -
0	1	1
1	0	1
1	1	0

Can also specify a boolean function using Boolean operations. Boolean "expression"

$$f(x,y) = \underline{\underline{x}\bar{y}} + \underline{\underline{\bar{x}}y}$$

$$x=y=0.$$

$$x=0 \quad y=1$$

$$f(x,y) = f(0,0) = 0$$

$$f(0,1) = 1.$$

A Boolean expression always has an equivalent input/output table.

Is the converse always true?

A literal for variable x is (x) or (\bar{x})

If f is a boolean function with input variables x_1, \dots, x_k

a minterm for f is a product of k literals in which each input variable or its negation (but not both) appears exactly once.

$$f(x, y, z)$$

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— — —

mintems:

- $x\bar{y}\bar{z}$
- $\bar{y}\bar{x}z$

not mintems:

$$\begin{aligned} &x\bar{y}\bar{z} \times \\ &\bar{x}\bar{y}\bar{z} \bar{x} \end{aligned} \quad \left. \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}$$
$$y\bar{z} \quad \text{---}$$
$$\overline{xyz}$$

A minterm evaluates to **1** for exactly one set of values for all the variables.

$$\rightarrow (\bar{x}\bar{y}\bar{z})$$

Evaluates to **1** if and only if

$$x=1, y=0, z=0.$$

$$x\bar{y}\bar{z}$$

$$1 \cdot 0 \cdot 1 = 1.$$

$$\begin{aligned} &x\bar{y}\bar{z} \\ &1 \cdot 0 \cdot 1 = 0 \end{aligned}$$

$$x=1, y=1, z=0.$$

$$f(x, y, z) = x\bar{y}\bar{z}$$

- Constructing a Boolean expression that computes the same function as defined in an input/output table.

<u>x</u>	<u>y</u>	<u>z</u>	<u>$f(x, y, z)$</u>
$\Rightarrow 0$	0	0	1 = $\bar{x}\bar{y}\bar{z}$
0	0	1	0
0	1	0	0
$\rightarrow 0$	1	1	1 = $\bar{x}yz$
1	0	0	0
1	0	1	0
$\rightarrow 1$	1	0	1 = $xy\bar{z}$
$\Rightarrow 1$	1	1	1 = xyz

$$f(x, y, z) = \underbrace{\bar{x}\bar{y}\bar{z}}_{\text{m}_0} + \underbrace{\bar{x}yz}_{\text{m}_1} + \underbrace{xy\bar{z}}_{\text{m}_2} + \underbrace{xyz}_{\text{m}_3}$$

$$= \underbrace{\bar{x}\bar{y}\bar{z}}_{\text{m}_0} + \underbrace{\bar{x}yz}_{\text{m}_1} + \underbrace{xy\bar{z}}_{\text{m}_2}$$

Boolean expressions can be put in standardized forms. (Useful for reasoning about them or manipulating them systematically).

Disjunctive Normal Form (DNF)

Sum of product of literals.

Congjunctive Normal Form (CNF)

Product of sums of literals.

Our method for converting I/O tables to Boolean expressions produces expressions in DNF form!

$$x\bar{y}z + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$$

The product terms need not be minterms.

The following Boolean expression are all in DNF form:

$$x\bar{y}z + \bar{x} + y\bar{z}$$

$$\underbrace{x(+)}_{\text{x is a product of}} y \oplus z$$

is a product of
one literal.

$$\bar{u}x\bar{z} + \bar{z}\bar{x} + y$$

$$\underbrace{xyz}_{\text{xyz}}$$

Literal: X or \bar{X}

A Boolean Expression is in DNF form iff

- (1) Multiplication only applied to literals.
- (2) Complement only applied to variables.

Not in DNF form:

$$(x+y)\bar{z} + \bar{x}\bar{y}\bar{z}$$

$\underbrace{\quad}_{\bar{x}} \quad \underbrace{\quad}_{z(x+y)} \quad \underbrace{\quad}_{\bar{x}}$

$$\bar{x}\bar{y} + \bar{y}\bar{z}$$

$\underbrace{\quad}_{\bar{x}\bar{y}} \quad \underbrace{\quad}_{\bar{y}\bar{z}}$

CNF Boolean expressions look like:

$$(\bar{x} + \bar{y} + \bar{z})(x + y)(\bar{u} + x)$$
$$x(\bar{y} + \bar{z})u$$
$$xy\bar{z}$$
$$(u + x + z)(y + \bar{x})$$

Both CNF & DNF.

A Boolean Expression is in CNF form iff

- (1) Addition only applied to literals.
- (2) Complement only applied to variables.

Not in CNF Form:

$$(x+y+uz)(\bar{x}+\bar{y})\bar{z}$$
$$xy + \bar{z}$$
$$(\bar{x}+\bar{y})(\bar{z}+\bar{y})$$

Does every Boolean function have a representation as a CNF Boolean expression?

Yes! Here is a systematic way for finding it:

Use a generalized version of De Morgan's Law:

$$\overline{(\bar{u} + \bar{v} + \bar{w} + \bar{x} + \bar{y} + \bar{z})} = \bar{\bar{u}} \cdot \bar{\bar{v}} \cdot \bar{\bar{w}} \cdot \bar{\bar{x}} \cdot \bar{\bar{y}} \cdot \bar{\bar{z}}$$

$$\text{Ex. } \neg(a \vee b \vee c) = \neg a \wedge \neg b \wedge \neg c.$$

$$\overline{uvwx\bar{y}\bar{z}} = \bar{\bar{u}} + \bar{\bar{v}} + \bar{\bar{w}} + \bar{\bar{x}} + \bar{\bar{y}} + \bar{\bar{z}}$$

<u>x</u>	<u>y</u>	<u>z</u>	<u>$f(x, y, z)$</u>	<u>$\bar{f}(x, y, z)$</u>
0	0	0	1	0
0	0	1	0	1 — $\bar{x}\bar{y}z$
0	1	0	0	1 — $\bar{x}yz$
0	1	1	1	0
1	0	0	0	1 — $\bar{x}\bar{y}\bar{z}$
1	0	1	0	1 — $\bar{x}\bar{y}z$
1	1	0	1	0
1	1	1	1	0

①

$$f(x, y, z) = \overline{f(x, y, z)} = \overline{\overline{xyz} + \overline{xy\bar{z}} + \overline{x\bar{y}\bar{z}} + \overline{x\bar{y}z}}$$

$$= \overline{A + B + C + D} = \overline{\bar{A} \bar{B} \bar{C} \bar{D}}$$

$$= (\overline{\bar{x}\bar{y}\bar{z}})(\overline{\bar{x}y\bar{z}})(\overline{x\bar{y}\bar{z}})(\overline{x\bar{y}z})$$

$$= (\bar{\bar{x}} + \bar{\bar{y}} + \bar{\bar{z}})(\bar{\bar{x}} + \bar{\bar{y}} + \bar{\bar{z}})(\bar{\bar{x}} + \bar{\bar{y}} + \bar{\bar{z}})(\bar{\bar{x}} + \bar{\bar{y}} + \bar{\bar{z}})$$

$$= (x + y + z)(x + y + z)(x + y + z)(x + y + z)$$



<u>x</u>	<u>y</u>	<u>$f(x, y)$</u>	<u>$\bar{f}(x, y)$</u>
0	0	1	0
0	1	1	0
1	0	1	0
1	1	0	1

$$\underline{\bar{f}(x, y)} = \underline{\bar{xy}} = \bar{x} + \bar{y}$$

Arbitrary boolean function
(expressed by input/output table) \rightarrow

Equivalent
Boolean expression
using only addition,
multiplication, complement
operations.

The set of operations : {addition, multiplication, complement}
if functionally complete.

Can we remove one of the operations + still remain
functionally complete.

$$\text{De Morgan} : \underline{xy} = \overline{\underline{x}\underline{y}} = \overline{\underline{x} + \underline{y}}$$

$$\underline{xy} = \underline{\overline{x}\overline{y}} = \underline{\overline{x} + \overline{y}}$$

Computes the product
without the multiplication op!

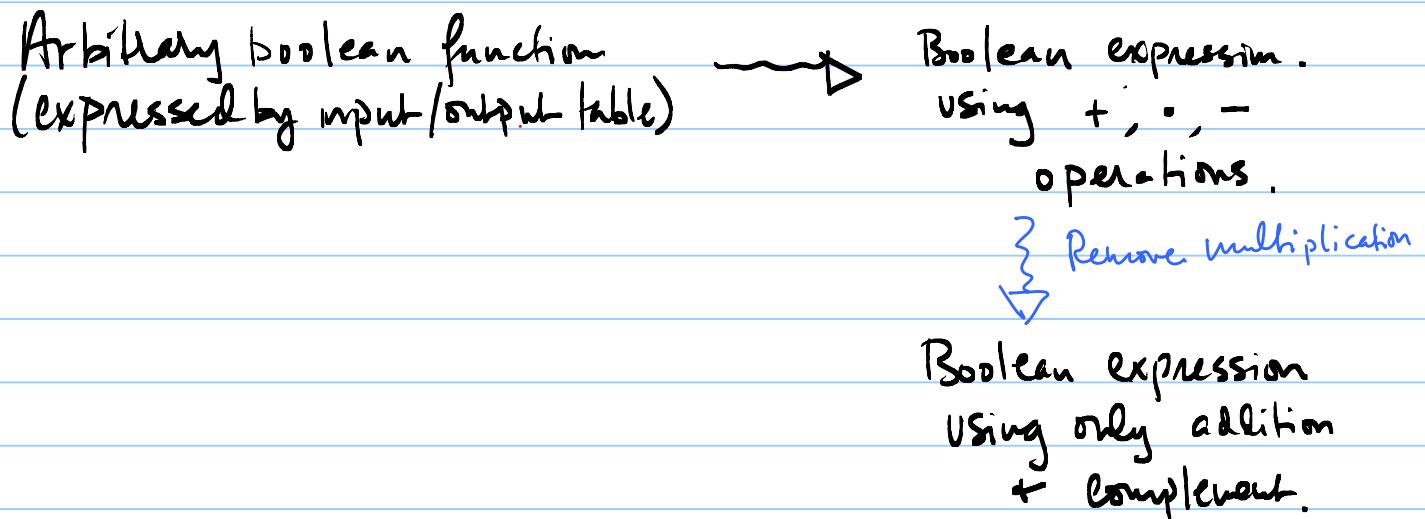
Apply within more complex expressions to
eliminate all multiplication operations.

$$\underline{(x+y)z} = \underline{(\overline{x}+\overline{y}) + \overline{z}}$$

$$(xy + z)w \quad (\overline{\underline{x}} + \overline{\underline{y}} + z)w$$

$$\underline{(\overline{\underline{x}} + \overline{\underline{y}} + z) + \overline{w}}$$

The set $\{\text{addition, complement}\}$ is functionally complete.



What about the set $\{\text{multiplication, complement}\}$?

De Morgan again!

$$\overline{x+y} = \overline{x}\overline{y}$$

$$x+y = \overline{\overline{x+y}} = \overline{\overline{x}\overline{y}}$$

Can use this rule to eliminate any addition operation in a Boolean expression.

$$(x+y)z \quad \overline{\overline{x}\overline{y}} \cdot z$$

$$\underline{xy + z} \quad \overline{\overline{x}\overline{y}\overline{z}}$$

$$xy + z(x+y)$$