

# Proof by Contradiction:

Note Title

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Theorem: If  $n$  is an integer and  $n^2$  is even, then  $n$  is also even.

•  $h_1$ :  $n$  is an integer.

$(h_1 \wedge h_2) \rightarrow c$ .

•  $h_2$ :  $n^2$  is even.

$c$ :  $n$  is even

$(h_1 \wedge \neg c) \rightarrow \neg h_2$

Proof Let  $n$  be an integer which is not even. We will show that  $n^2$  is odd.

Since  $n$  is an integer and not even then  $n$  is odd.

Therefore  $n = 2k+1$  for some integer  $k$ . Therefore,

$$\begin{aligned} n^2 &= (2k+1)^2 = 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Since  $k$  is an integer  $2k^2 + 2k$  is also an integer. Therefore  $n^2$  is odd.  $\square$

Why contrapositive?

$n^2$  is even.

$n^2 = 2k$  integer  $k$ .

$n = \sqrt{2k}$

Proof by contradiction.

Theorem:  $t$ .

Assume  $\neg t$

Show that this leads to a contradiction:

$r \wedge \neg r$ .

Can be used to prove theorems that are not conditional statements.

Direct and Contrapositive proofs specifically for theorems of the form  $p \rightarrow c$ .

Proof by contrapositive a special case of a  $t \equiv (p \rightarrow c)$  proof by contradiction.

Assume  $\neg c \wedge p$   
show this leads to  $\neg p$ .

Contradiction:  $\neg p \wedge p$ .

$$\begin{aligned}(p \rightarrow c) &\equiv (\neg p \vee c) \\ \underline{\neg(p \rightarrow c)} &\equiv \neg(\neg p \vee c) \\ &\equiv \neg\neg p \wedge \neg c \\ &\equiv p \wedge \neg c.\end{aligned}$$

Theorem  $\sqrt{2}$  is an irrational number.

Proof: Proof by contradiction. Assume  $\sqrt{2}$  is rational.

$\sqrt{2} = \frac{n}{d}$  where  $n/d$  is in reduced form. (there is no integer  $> 1$  that divides  $n + d$ ).

Square both sides to get.  $(\sqrt{2})^2 = \left(\frac{n}{d}\right)^2$

$$2 = \frac{n^2}{d^2}$$

Multiply both sides by  $d^2$ :  $2d^2 = n^2$

Since  $n$  is an integer +  $n^2$  is even, then  $n$  is even.

$n = 2k$  for some integer  $k$ .

$$2d^2 = n^2 = (2k)^2 = 2^2 \cdot k^2 = 4k^2$$

Divide both sides by 2:  $d^2 = 2k^2$ .

$d$  is an integer and  $d^2$  is even. Therefore  $d$  is even.

$n + d$  are both even, so 2 divides both  $n + d$ . Which contradicts the fact that  $n + d$  have no common divisors  $> 1$ . Therefore  $\sqrt{2}$  is irrational.  $\square$

- \* Integer  $p$  is prime if  $p > 1$  and no positive integers divide  $p$  besides  $1$  &  $p$ .
- \* If positive integer  $n$  is not prime, then there is a prime  $p$  that evenly divides  $n$ .

Theorem: There are an infinite number of prime numbers.

Proof: Proof by contradiction.

Suppose that there are a finite number of primes:  $p_1, p_2, \dots, p_k$ .

Consider the integer:

$$(p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_k) + 1 = N.$$

$N$  is an integer and larger than all the primes and therefore  $N$  is not prime.

At least one of the primes is a divisor of  $N$ .

Suppose  $p_j$  evenly divides  $N$ . Then  $N/p_j$  is an integer.

$$1 = N - (p_1 p_2 p_3 \dots p_k)$$

Divide by  $p_j$ :

$$\frac{1}{p_j} = \frac{N}{p_j} - \frac{p_1 p_2 \dots p_j \dots p_k}{p_j}$$

not an integer. ↑ int. integer.

The right side of the equation is an integer, but  $p_j > 1$  so  $1/p_j < 1$  and  $1/p_j$  can not be an integer. This is a contradiction and therefore there must be an infinite # of primes.  $\square$

Theorem There is no smallest integer.

Proof By contradiction.

Let  $x$  be the smallest integer.

If  $x$  is an integer, then  $x-1$  is also an integer.

Add  $x$  to  
both sides:

$$\begin{array}{r} -1 < 0 \\ \hline x-1 < 0+x = x \end{array}$$

Since  $x-1$  is an integer that is smaller than  $x$ , then  $x$  can not be the smallest integer.  $\square$