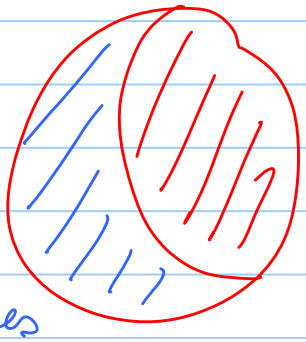
Proofs by Cases

Note Title 1/27/2015

Sometimes hard to prove a statement $\forall x P(x)$ for all x in the domain at the same time.



Idea: Break up the domain into different classes and prove the theorem for each class separately.

Need to make sure all the elements in the domain are covered in the proof.

Theorem If x and y are integers with the same parity, then $x+y$ is even.

Proof Case 1: x and y are odd.

Then $x = 2k+1$

$y = 2k'+1$ for integers k & k'.

Then $x+y = (2k+1)+(2k'+1) = (2k+2k'+2)$

$= 2(k+k'+1)$.

Case 2: x and y are even.

$x = 2k$ and $y = 2k'$ for ints k & k'.

$x+y = 2k+2k' = 2(k+k')$.

Theorem: For every integer x, $|x| \leq x^2$

Definition:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0. \end{cases}$$

$|-3| = 3 = -(-3)$

positive x:

$|x| \leq x^2$

$x \leq x^2$

$\frac{x}{x} \leq \frac{x^2}{x}$

$1 \leq x.$

proof.

negative x:

$x < 0$

$-x > 0$

$|x| \leq x^2$

$-x \leq x^2$

$\frac{-x}{-x} \leq \frac{x^2}{-x}$

$1 \leq -x$

Theorem: For every integer x, $|x| \leq x^2$

Proof Case 1: $x = 0$. $|0| = 0 \leq 0^2$.

Case 2: $x < 0$. Then $|x| = -x$ and $-x > 0$.
Since $-x$ is an integer, $1 \leq -x$. Multiply inequality by $-x$ to get:
$$-x \leq (-x) \cdot (-x)$$
Since $-x = |x|$ and $(-x)^2 = x^2$, then $|x| \leq x^2$.

Case 3: $x > 0$. Then $|x| = x$. Since x is an integer and $x > 0$, then $1 \leq x$.
Multiply both sides of the inequality by x:
$$x \leq x \cdot x.$$
Since $x = |x|$, then $|x| \leq x^2$. ☒.

Sometimes a proof by contrapositive results in a proof by cases:

Theorem: If x and y are real numbers and $xy \neq 0$ then $x \neq 0$ and $y \neq 0$.

Domain: real numbers.

$$\forall x \forall y \; ((xy \neq 0) \rightarrow (x \neq 0 \wedge y \neq 0))$$

$$\equiv \forall x \forall y \; (\neg(x \neq 0 \wedge y \neq 0) \rightarrow \neg(xy \neq 0))$$

$$(\neg(x \neq 0) \vee \neg(y \neq 0) \rightarrow (xy = 0))$$

$$((x = 0 \text{ OR } y = 0) \rightarrow (xy = 0))$$

Proof: Proof by contrapositive. Assume that for real numbers, x and y, $x = 0$ or $y = 0$. We will show that $xy = 0$.

Case 1: $x = 0$. $\quad x \cdot y = 0 \cdot y = 0$.

Case 2: $y = 0$. $\quad x \cdot y = x \cdot 0 = 0$. ☒

It's ok if the cases overlap. The scenario $x = y = 0$ falls under case 1 and case 2.

Theorem: If x and y are real numbers and $xy = 0$, then $x = 0$ or $y = 0$.

Domain: real numbers.

$$\forall x \forall y \left((xy = 0) \rightarrow (x = 0 \vee y = 0) \right)$$

$$\forall x \forall y \left(\neg (x = 0 \vee y = 0) \rightarrow \neg (xy = 0) \right)$$

$$\left((\neg (x = 0) \wedge \neg (y = 0)) \rightarrow xy \neq 0 \right)$$

$$\left(((x \neq 0) \wedge (y \neq 0)) \rightarrow xy \neq 0 \right).$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p.$$