Student ID Number:

Name:____________________________________

Test I

Version A

ICS 6D
Fall 2016
Oct 21, 2016
Instructor: Sandy Irani

Instructions

- Wait until instructed to turn over the cover page.
- The total number of points on the test is 42.
- **Important:** Except for the cover page, there are questions on both sides of the page.

1. *(4 points)* A set S is a subset of $\{a, c\}^*$ and is defined recursively as follows:

   Basis: $\lambda \in S$ and $a \in S$.

   Recursive rules: If string x is in S, then

   - $axc \in S$
   - $cxa \in S$

   Exclusion statement: A string x is in S only if x is given in the basis or x can be constructed by repeatedly applying the recursive rule starting with a string given in the basis.

   (a) List all the strings of length 3 that are in S.

   (b) List all the strings of length 4 that are in S.

2. *(2 points)* Using summation notation, express the sum of the cubes of the even integers between 1 and 201.

3. *(2 points)* Write down an equivalent expression to the summation below where the last term is outside the summation:

$$\sum_{j=-2}^{n-2} j \cdot 2^{j+1}$$

4. *(4 points)* Let $S(n)$ be a statement parameterized by a positive integer n. Consider a proof that uses strong induction to prove that for all $n \geq 3$, $S(n)$ is true. If the base case proves that $S(3)$, $S(4)$, $S(5)$ and $S(6)$ are all true, then what would be the inductice hypothesis in the proof of the inductive step?

5. *(2 points)* A sequence $\{a_n\}$ is defined as follows: $a_0 = 2$, $a_1 = 1$. For $n \geq 2$:

$$a_n = 3a_{n-1} - n \cdot a_{n-2} + 1$$

   What is a_3?

6. *(8 points)* The sequence $\{g_n\}$ is defined recursively as follows: $g_0 = 1$. For $n \geq 1$, $g_n = 3 \cdot g_{n-1} + 2n$.

   Theorem 1. *For any non-negative integer n, $g_n = (5/2) \cdot 3^n - n - (3/2)$.*

   Below is an inductive proof of the theorem with some lines missing. Fill in the blanks to complete the proof.

   Proof:

   Bases Case: ____

   Inductive Step:

   For $k \geq$ __________, assume ____,

   and prove: $g_{k+1} = (5/2) \cdot 3^{k+1} - (k+1) - (3/2)$.

   g_{k+1} = ____ (by definition)

   = ____ (by the inductive hypothesis)

$$
\begin{aligned}
&= (5/2) \cdot 3 \cdot 3^k - 3k - 3 \cdot (3/2) + 2k + 2 \\
&= (5/2) \cdot 3^{k+1} - k - (5/2) \\
&= (5/2) \cdot 3^{k+1} - (k+1) - (3/2)
\end{aligned}
$$

■

7. *(8 points)* A sequence $\{f_n\}$ is defined by the following recurrence relations and initial conditions.

- $f_0 = 3$
- $f_1 = 4$
- $f_n = 8 \cdot f_{n-1} - 16 \cdot f_{n-2}$

(a) Give the characteristic equation for the recurrence relation.

(b) Give the general solution for the recurrence relation.

(c) Write the linear equations that need to be solved in order to find the solution to the recurrence relation and initial conditions.

(d) Give the solution to the recurrence relation and initial conditions.

8. *(8 points)* A function receives a positive integer n as the input and returns the value

$$\text{ComputeSum}(n) = \sum_{j=1}^{n} (j+2)j^2.$$

Fill in the blanks for recursive algorithm to compute the sum given above. The value returned should be a mathematical expression that uses variables y and/or n and only uses addition or multiplication operations. There should be no summation notation in your responses.

```
ComputeSum(n)

    If ( ____________ ) Return( ____________ )      // Base case

    y := ComputeSum(____________) // Recursive Call

    Return( ____________________________ )
End
```

9. *(4 points)* The function below receives two inputs: a and n, where a is a real number and n is a non-negative integer. The algorithm returns a times n.

```
Product( a, n )

    If ( n = 0 ) Return( 0 )

    y := Product( a, n − 1 )

    Return( a + y )
End
```

Let $\text{NumAdd}(a, n)$ be the number of addition operations performed by the algorithm Product on inputs a and n.

(a) What is $\text{NumAdd}(a, 0)$?

(b) Express $\text{NumAdd}(a, k+1)$ is a function of $\text{NumAdd}(a, k)$.