Tensor Products

Suppose that we have two quantum systems with Hilbert spaces:

- \( V \): \( \mathbb{C}^k \) (k distinguishable states)
- \( W \): \( \mathbb{C}^l \) (l distinguishable states)

What is the Hilbert space for the composite system? (Note that the composite system has \( k \cdot l \) distinguishable states.)

**Hilbert Space** \( V \otimes W \) (*V tensor W*)

Suppose we have:

- Basis for \( V \): \( |v_1\rangle, |v_2\rangle, \ldots, |v_k\rangle \)
- Basis for \( W \): \( |w_1\rangle, |w_2\rangle, \ldots, |w_l\rangle \)

A basis for \( V \otimes W \) will be:

\[ |v_i\rangle \otimes |v_j\rangle \quad 1 \leq i \leq k \]
\[ 1 \leq j \leq l \]

An arbitrary state \( |\phi\rangle \) in \( V \otimes W \) can be expressed as:

\[ |\phi\rangle = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq l} d_{ij} |v_i\rangle \otimes |w_j\rangle \]

Inner product extends to states in \( V \otimes W \) as:

\[ (|v_i\rangle \otimes |w_i\rangle, |v_j\rangle \otimes |w_j\rangle) = \langle v_i | v_j \rangle \langle w_i | w_j \rangle \]

This is extendable to arbitrary states by linearity.

We have already seen examples of tensor products when we looked at systems of n qubits.

**Hilbert Space for one qubit**: \( \mathbb{C}^2 \)

**Hilbert Space for n-qubits**: \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \)
Note that when we add a qubit, we double the number of complex numbers required to specify an arbitrary state.

We can define linear operators on a composite system by taking the tensor product of two operators, each acting on a subsystem... an linear operators on those.

\[ A_V \text{ acts on } V \quad (A_V \otimes A_W) |v\rangle \otimes |w\rangle = \]
\[ A_W \text{ acts on } W \quad A_V |v\rangle \otimes A_W |w\rangle \]

The action of \( A_V \otimes A_W \) on an arbitrary state in \( V \otimes W \) is defined by linearity.

For example:

- \( V = \mathbb{C}^2 \) (one qubit) basis: \( \{ |0\rangle, |1\rangle \} \)
- \( W = \mathbb{C}^2 \) (another qubit) basis: \( \{ |0\rangle, |1\rangle \} \)

\[ V \otimes W = \mathbb{C}^2 \otimes \mathbb{C}^2 \] Basis for \( V \otimes W \):
- \( |0\rangle \otimes |0\rangle = |00\rangle \)
- \( |0\rangle \otimes |1\rangle = |01\rangle \)
- \( |1\rangle \otimes |0\rangle = |10\rangle \)
- \( |1\rangle \otimes |1\rangle = |11\rangle \)

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ 10\langle 0 | - |1\rangle\langle 1 | \]

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ |0\rangle\langle 1 | + |1\rangle\langle 0 | \]

\[ (Z \otimes X) |10\rangle = Z |1\rangle \otimes X |0\rangle = - |11\rangle \]

\[ - |1\rangle \quad |1\rangle \]
What does the matrix representation of $A_v \otimes A_w$ look like? Pick a basis for $V + W$. Express $A_v + A_w$ in matrix form $(M_v + M_w)$

$M_v \otimes M_w$ is a $k_l \times k_l$ matrix

$$(M_v \otimes M_w)_{i_1 k + j_1, i_2 k + j_2} = [M_v]_{i_1 i_2} \cdot [M_w]_{j_1 j_2}.$$ 

$k \times k$ blocks, each of size $l \times l$.

In our example above

$$2 \otimes x = \begin{bmatrix} 1 & [x] & 0 & [x] \\ 0 & [x] & -1 & [x] \end{bmatrix} = \begin{bmatrix} \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} & \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \end{bmatrix}.$$ 

We will use this a lot when we talk about gates and circuits. If we have $n$-qubits on which we are performing a computation, we typically only operate on 2 qubits at a time. Thus, a gate is specified by a $4 \times 4$ matrix but it is really operating on a big Hilbert space of dimension $2^n$, so any operator must be a $2^n \times 2^n$ matrix. The gate matrix is really tensored
with the identity on the rest of the qubits.

A 2-qubit gate we will encounter a lot is a CNOT gate

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

denoted by:

\[\text{\textbullet} - \text{\textbullet}\]

\(q_1\) is the control bit if \(q_1 = 0\), \(q_2\) is unchanged

\(q_2\) is the target bit if \(q_1 = 1\), \(q_2\) is flipped

if we have

\[\text{\textbullet} - \text{\textbullet}\]

This is really:

\[\text{[CNOT]}_{12} \otimes I_{3\ldots n}\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\(2^{n-2} \times 2^{n-2}\)

all 0's

\[\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\(2^{n-2} \times 2^{n-2}\)

identity

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

If instead we had \(I_{1\ldots n-2} \otimes [\text{CNOT}]_{n-1,n}\) we would have:

\[\text{\textbullet} - \text{\textbullet}\]

\[\begin{bmatrix}
\begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Here are some examples of 1-qubit gates that we will frequently encounter:

\[ H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]  
thus flips between 0/1 + + bases:

\[ H|0\rangle = |+\rangle \quad H|+\rangle = |0\rangle \]
\[ H|1\rangle = |-\rangle \quad H|-\rangle = |1\rangle \]

\[ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ phase flip} \]
\[ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ not} \]

Rotate by \( \theta \):

\[ U_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]
(for real-valued amplitudes)

\[ |0\rangle \rightarrow U_\theta (|0\rangle) = \cos \theta |0\rangle + \sin \theta |1\rangle \]

\[ |1\rangle \rightarrow U_\theta (|1\rangle) = \cos \theta |1\rangle + \sin \theta |0\rangle \]

Tensor Product for States:

\[ |\psi_v\rangle = \sum_{i=1}^{k} \alpha_i |v_i\rangle \quad |\psi_w\rangle = \sum_{j=1}^{l} \beta_j |w_j\rangle \]

\[ |\psi_v\rangle \otimes |\psi_w\rangle = \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_i \beta_j |v_i\rangle \otimes |w_j\rangle \]

For example, suppose we have 2 qubits:

\[ |\psi_1\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \]
\[ |\psi_2\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle \]

\[ |\psi_1\rangle \otimes |\psi_2\rangle = \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle \]

Not every state on 2 qubits can be expressed as a tensor product:

\[ \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]  
entangled state

This has important implications for measurement.
In the case $|1\rangle_1 \otimes |1\rangle_2$ if we measure first qubit if the outcome is 0 state afterwards is $|0\rangle_1 |1\rangle_2$ if the outcome is 1 state afterwards is $|1\rangle_1 |1\rangle_2$

Measuring qubit 1 gives us no information about qubit 2.

Now what if we measure the first qubit of the entangled state $\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$.

- Outcome is 0 w.p. $\frac{1}{2}$
- Outcome is 1 w.p. $\frac{1}{2}$

After measuring the first qubit we know the value of the second qubit with certainty (measured in standard basis). - even if qubits are physically separated