

## Homework 3

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Sections 7.1, 7.2, 7.4, 10.3

In each of the inductive proofs below, you must first state that you are proving the theorem by induction. The base case and inductive step must be clearly labeled. At the beginning of the inductive step, you need to state clearly what you are assuming and what you are proving. You must also clearly indicate where you are using the inductive hypothesis.

1. Prove that any amount of postage with 12 cents or more can be made from 5-cent or 4-cent stamps.
2. The sequence  $\{g_n\}$  is defined recursively as follows:
  - $g_0 = 51$
  - $g_1 = 348$ .
  - $g_n = 5 \cdot g_{n-1} - 6 \cdot g_{n-2} + 20 \cdot 7^n$ , for  $n \geq 2$ .

Use induction to show that  $g_n = 3^n + 2^n + 49 \cdot 7^n$ .

3. Give a recursive definition for strings of properly nested parentheses and curly braces. For example  $\{\}\{\}\{\}$  is properly nested but  $\{\}\{\}$  is not.
4. Give an inductive proof that for sequences in the set defined in the previous problem, the number of left regular parens ( is the same as the number of right regular parens ) and the number of left curly braces { is equal to the number of right curly braces }.
5. Let  $B = \{a, b\}$ .
  - (a) Give a recursive definition for  $B^*$ .
  - (b) Give a recursive definition for the set  $S$  which is all strings from  $B^*$  such that all the  $a$ 's come before all the  $b$ 's. For example  $aabbbb \in S$ , but  $aababbbb \notin S$ .
6. Give a recursive algorithm that takes as input two positive integers  $x$  and  $y$  and returns the product of  $x$  and  $y$ . The only arithmetic operations your algorithm can perform are addition or subtraction. Furthermore, your algorithm should have no loops.
7. Give a recursive algorithm that takes as input a non-negative integer  $n$  and returns a set containing all binary strings of length  $n$ . Here are the operations on strings and sets you can use:
  - Initialize an empty set  $S$ .
  - Add a string  $x$  to  $S$ .
  - $y := 0x$  (This operation adds a 0 to the beginning of string  $x$  and assigns the result to string  $y$ ).
  - $y := 1x$  (This operation adds a 1 to the beginning of string  $x$  and assigns the result to string  $y$ ).
  - return( $S$ )

Also, you can have a looping structure that performs an operation on every string in a set:

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For every x in S
    //perform some steps with string x
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8. Give a recursive algorithm whose input is  $a$ , a real number and integer  $n$  such that  $n \geq 0$ , and whose output is  $a^{2^n}$ . **Note that the exponent of  $a$  is  $2^n$ .**
9. Use induction to prove that your algorithm in the previous question returns the correct result.
10. Give the characteristic equation for the following recurrence relations:
  - (a)  $a_n = 3a_{n-1} - a_{n-2} + 17a_{n-3}$ .
  - (b)  $a_n = -a_{n-2} + a_{n-4}$
  - (c)  $a_n = 3a_{n-7}$
11. Solve the following recurrence relation:
  - $f_n = f_{n-1} + 12f_{n-2}$
  - $f_0 = -2$
  - $f_1 = 20$
12. Solve the following recurrence relation:
  - $f_n = 3f_{n-1} + 4f_{n-2}$
  - $f_0 = 4$
  - $f_1 = 1$
13. Solve the following recurrence relation:
  - $f_n = 4f_{n-1} - 4f_{n-2}$
  - $f_0 = 3$
  - $f_1 = -10$

The following problems are for practice only. They will not be graded.

1.
  - $f_0 = \frac{5}{3}$
  - $f_1 = \frac{11}{3}$ .
  - $f_n = 3 \cdot f_{n-1} + 4 \cdot f_{n-2} + 6n$ , for  $n \geq 2$ .

Use induction to show that  $f_n = 2 \cdot 4^n + \frac{3}{2}(-1)^n - n - \frac{11}{6}$ .
2. Solve the following recurrence relation:
  - $f_n = 4f_{n-2}$
  - $f_0 = 2$
  - $f_1 = 16$
3. Solve the following recurrence relation:
  - $f_n = 3f_{n-1} + 4f_{n-2} - 12f_{n-3}$
  - $f_0 = 4$
  - $f_1 = -5$
  - $f_2 = 11$
4. Solve the following recurrence relation:
  - $f_n = 5f_{n-1} - 8f_{n-2} + 4f_{n-3}$
  - $f_0 = 6$
  - $f_1 = 7$
  - $f_2 = 17$