Note Title 1/27/2015

Proof by Contradiction:

Theorem: If n is an integer and n^2 is even, then n is also even.

h_1: n is an integer.
h_2: n^2 is even.
c: n is even

$(h_1 \wedge h_2) \to c$.

$(h_1 \wedge \neg c) \to \neg h_2$.

Proof Let n be an integer which is not even. We will show that n^2 is odd:

Since n is an integer and not even then n is odd.

Therefore $n = 2k+1$ for some integer k.

$$n^2 = (2k+1)^2 = 4k^2+4k+1$$
$$= 2(2k^2+2k)+1$$

Since k is an integer $2k^2+2k$ is also an integer. Therefore n^2 is odd.

$\boxtimes$

Why contrapositive? n^2 is even.

$n^2 = 2k$ integer k.

$n = \sqrt{2k}$

Proof by contradiction.

Theorem: t.

Assume $\neg t$

Show that this leads to a contradiction:

$r \wedge \neg r$.

Can be used to prove theorems that are not conditional statements.

Direct and Contrapositive proofs specifically for theorems of the form $p \to c$.

$\neg c \to \neg p$.

Proof by contrapositive a special case of a proof by contradiction.

Assume $\neg c \wedge p$

show this leads to $\neg p$.

Contradiction: $\neg p \wedge p$.

$(p \to c) \equiv (\neg p \vee c)$

$\Rightarrow \neg(p \to c) \equiv \neg(\neg p \vee c)$.

$\equiv \neg\neg p \wedge \neg c$.

$\equiv p \wedge \neg c$.

Theorem There is no smallest integer.

Proof By contradiction.

Let x be the smallest integer.

If x is an integer, then $x-1$ is also an integer.

$$-1 < 0$$

Add x to both sides:

$$x-1 < 0+x = x$$

$$x-1 < x.$$

Since $x-1$ is an integer that is smaller than x, then x can not be the smallest integer.

Theorem $\sqrt{2}$ is an irrational number.

Proof: Proof by contradiction. Assume $\sqrt{2}$ is rational.

$\sqrt{2} = \frac{n}{d}$ where n/d is in reduced form. (there is no integer > 1 that divides n + d).

Square both sides to get. $(\sqrt{2})^2 = \left(\frac{n}{d}\right)^2$

$$2 = \frac{n^2}{d^2}.$$

Multiply both sides by d^2: $2d^2 = n^2$

Since n is an integer + n^2 is even, then n is even.

$n = 2k$ for some integer k.

$$2d^2 = n^2 = (2k)^2 = 2^2 \cdot k^2 = 4k^2$$

Divide both sides by 2: $d^2 = 2k^2$.

d is an integer and d^2 is even. Therefore d is even.

n + d are both even, so 2 divides both n + d. which contradicts the fact that n + d have no common divisors > 1. Therefore $\sqrt{2}$ is irrational. ☒

[* Integer p is prime if $p > 1$ and no positive integers divide p besides 1 & p.

* If positive integer n is not prime, then there is a prime p than evenly divides n.]

Theorem: There are an infinite number of prime numbers.

Proof: Proof by contradiction.
Suppose that there are a finite number of primes: $P_1, P_2, \ldots, P_k$.

Consider the integer:

$$(P_1 \cdot P_2 \cdot P_3 \cdot \cdots \cdot P_k) + 1. = N.$$

N is an integer and larger than all the primes and therefor N is not prime.

At least one of the primes is a divisor of N.

Suppose P_j evenly diveds N. Then N/P_j is an integer.

$$1 = N - (\underline{P_1 P_2 P_3 \cdots P_k})$$

Divide by P_j:
$$\frac{1}{P_j} = \underbrace{\frac{N}{P_j}}_{\text{int}} - \underbrace{\frac{P_1 P_2 \cdots \cancel{P_j} \cdots P_k}{\cancel{P_j}}}$$

The right side of the equation is an integer, but $P_j > 1$ so $1/P_j < 1$ and $1/P_j$ can not be an integer. This is a contradiction and therefor there must be an infinite # of primes. ☒

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Proofs by Cases

Sometimes hard to prove a statement $\forall x P(x)$ for all x in the domain at the same time.

Idea: Break up the domain into different classes and prove the theorem for each class separately.

Need to make sure all the elements in the domain are covered in the proof.

Theorem If x and y are integers with the same parity, then $x+y$ is even.

Proof Case 1: x and y are odd.

Then $x = 2k+1$
$y = 2k'+1$ for integers k & k'.

$$x+y = (2k+1) + (2k'+1) = 2k+2k'+2$$
$$= 2(k+k'+1)$$

$\therefore$ $x+y$ is even.

Case 2: x and y are even.

$x = 2k$
$y = 2k'$ for ints k & k'.

$$x+y = 2k + 2k' = 2(k+k')$$

$\therefore$ $x+y$ is even.

A number is a perfect square if it is equal to n^2, where n is an integer.

Theorem: Every perfect square is a multiple of 4 or a multiple of 4 plus 1.

$\rightarrow 4k.$

$\rightarrow 4k+1$

Proof Let n be an integer. We will show that n^2 is a multiple of 4 or a multiple of 4 plus 1.

implicit use of universal generalization.

Case 1: n is even. Then $n = 2k$ for some integer k.

$$n^2 = (2k)^2 = 4k^2$$

$\therefore$ n^2 is a mult of 4.

Case 2: n is odd. So $n = \underline{2k+1}$ for some int k.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$= 4(k^2+k) + 1$$

So n^2 is a mult of 4 plus 1.

More Examples of Proofs

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$x=2$ $\quad -\cancel{8}+\cancel{8}+1>0.$

Theorem If $0 \le x \le 2$ then $-x^3+4x+1>0$.

Scratch $x=0$. $\quad 1>0$ ✓ $\qquad x=1$. $\quad -1+4+1>0$

For x in the range from 0 to 2 $4x$ is getting larger faster than x^3.

Try factoring: $-x^3+4x = x(4-x^2) = x(2-x)(2+x)$

for $0 \le x \le 2$ all factors ≥ 0

$\Rightarrow$ Product is ≥ 0.

If $-x^3+4x \ge 0$ then $-x^3+4x+1 \ge 1 > 0$.

Proof: Suppose $0 \le x \le 2$.
Then $\boxed{x \ge 0}$

By adding 2 to both sides:
$x+2 \ge 2$, so $\boxed{2+x \ge 0.}$

Since $2 \ge x$, by subtracting x from both sides: $2-x \ge x-x \ge 0$, and $\boxed{2-x \ge 0.}$

Therefore when x satisfies $0 \le x \le 2$, then x, $2+x$, and $2-x$ are all at least 0.

Therefore $x\cdot(2+x)(2-x) \ge 0$.
Multiplying out, we get $4x - x^3 \ge 0$.
Add 1 to both sides and $-x^3+4x+1 \ge 1 > 0$.
and therefore $x^3-4x+1>0$.

Theorem: If 5 is not a divisor of xy for integers x & y, then 5 is not a divisor of x and 5 is not a divisor of y.

$$\underbrace{(5 \text{ not div of } xy)}^{h} \rightarrow \overbrace{[5 \text{ not div of } x \wedge 5 \text{ not div. of } y]}^{c}$$

$\neg c$

$$\neg \overbrace{[5 \text{ not divisor of } x \wedge 5 \text{ not divisor of } y]}^{c}$$

$$\neg(5 \text{ not div of } x) \vee \neg(5 \text{ not div of } y)$$

$$5 \text{ is a divisor of } x \vee 5 \text{ is a divisor of } y$$

$$(x = 5k \text{ for some int } k) \vee (y = 5k' \text{ for some int } k')$$

$\Rightarrow$ Cases!

Proof: Assume that 5 is a divisor of x or 5 is a divisor of y. Then we will show that 5 is a divisor of xy.

Case 1: 5 is a divisor of x. Therefore $5k = x$ for some integer k.

Then $xy = 5ky$
and 5 is also a divisor of xy.

Case: 5 is a divisor of y. Therefore $5k' = y$ for some integer k'.

Then $xy = x \cdot 5k' = 5xk'$
and 5 is also a divisor of xy. $\boxtimes$