Functions

Note Title 10/10/2014

Continuous functions: $f(x) = x^2 - 3$
$f(x) = \sin^2(x)$

Discrete math: functions whose inputs + outputs are from finite or countably infinite sets.

Example: map a set of computational tasks to computers in a distributed network.

The input/output relationship of a digital circuit. (or computer program)

First part of definition: input + output sets.

$f: A \to B$

A is the domain
B is the target } A + B are sets.
co-domain.

$f: \{0,1\}^5 \to \{0,1\}^4$

$f(01101) = 1001$

input: binary strings length 5
output: binary strings of length 4.

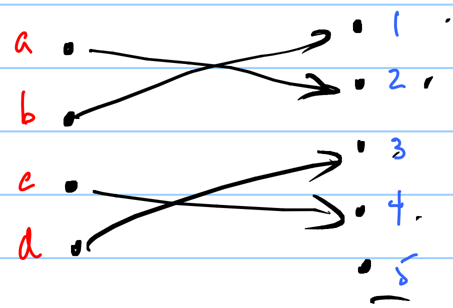
f must be well-defined for each element in the domain.
one output in range for each input.

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = 1/x$ Not defined for $x=0$.

$f(x) = \pm\sqrt{x^2+1}$ Two different outputs for each input.

Arrow diagram. $X = \{a, b, c, d\}$. $Y = \{1, 2, 3, 4, 5\}$. $g : X \to Y$



$f(a) = 2$
$f(b) = 1$
$f(c) = 4$
$f(d) = 3$

Range of a function $f \subseteq$ Target.

Range of $f = \{ y \in Y : \exists x \in X \ f(x) = y \}$.

For example above: $\{1, 2, 3, 4\}$

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad$ range: non-neg reals.

$g : \underline{\mathbb{Z} \to \mathbb{Z}}$ $\quad f(x) = 3x$ $\quad$ range: multiples of 3

$f : \underline{\{0,1\}^2} \to \{0,1\}^3$ $\quad f(x)$: copy the first bit and append to the end.

$\{00, 01, 10, 11\}$.

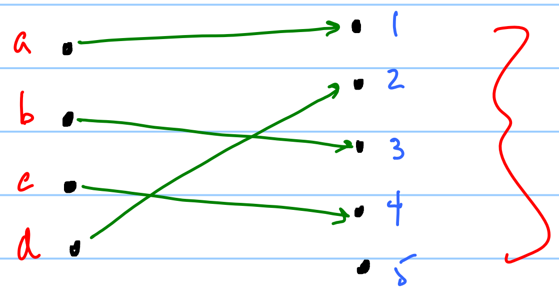
$f(00) = 000$.
$f(01) = 010$.
$f(10) = 101$.
$f(11) = 111$

Range: $\{000, 010, 101, 111\}$.

Range = Target? NO

$\Rightarrow$ 001 in target but not the range.

A function $f: X \to Y$ is <u>onto</u> if the range of f = Y.
$\forall y \in Y \ \exists x \in X \ f(x) = y$.



onto?

Note: if $|Y| > |X|$ then f can not be onto.
if f is onto then $|Y| \leq |X|$.

$f: \{0,1\}^3 \to \{0,1\}^3$

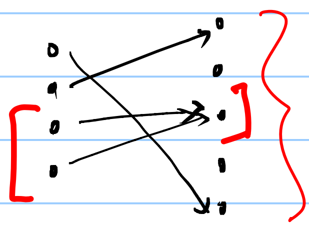
cyclic shift left
Remove 1st bit & put it at the end.

$f(101) = 011$
$f(011) = 110$

onto?

A function $f: X \to Y$ is 1-1 (one-to-one) if no two elements in the domain map on to same element in the target.

$x, x' \in X \quad x \neq x' \Rightarrow f(x) \neq f(x')$.



not 1-1.

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$f(3) = f(-3) = 9$.

Recap: $f: \underset{\text{set of}}{\text{Employees}} \to \underset{\text{set of}}{\text{Offices}}$ f: Office assignment.

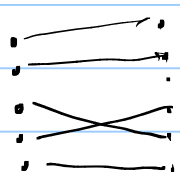
e.g. $f(\text{Sam}) = \text{Rm } 112$.

If f is onto then no office is unoccupied.

If f is 1-1 then everyone gets their own office.

$g: \mathbb{Z} \to \mathbb{Z}$ $g(x) = x^2$ not 1-1. $g(2) = g(-2) = 4$.

If $g: X \to Y$ is 1-1 then $|Y| \geq |X|$.



$f: \underline{\{0,1\}^2} \to \{0,1\}^2$ $\underset{b_1 b_2}{f(x)} = \underbrace{b_1 \cdot b_2}, \underbrace{0}$.

$f(01) = 00$ 1
$f(10) = 00$
$f(11) = 10$
$f(00) = 00$

not 1-1! not onto! 01, 11 not in range.

$g: \mathbb{Z} \to \mathbb{Z}$ $g(x) = x+2$. both. what if $g: \mathbb{N} \to \mathbb{N}$? not onto, 1-1

$= 2x$. 1-1, not onto.

$= x/2$ (integer division, throw away remainder) not 1-1 $g(1) = g(0) = 0$.

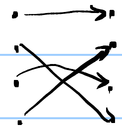
$x+2 = y+2 \Downarrow x = y$. $x+2 \neq y+2 \Uparrow x \neq y$

onto $g(2y) = y$.

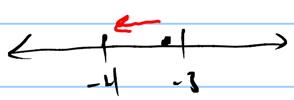
If g is 1-1 and onto it is a bijection.
(one-to-one correspondence).

If $g: X \to Y$ is one-to-one $\Rightarrow |Y| \geq |X|$.
onto $\Rightarrow |X| \geq |Y|$ $\Big\}$ $|X| = |Y|$

If $|X| \neq |Y|$ then f is not a bijection.

f:  pairs up elements from domain + target.

floor : $\mathbb{R} \to \mathbb{Z}$ $\quad \lfloor x \rfloor$ (round down) $\rightsquigarrow$ largest int $\leq x$.
ceiling : $\mathbb{R} \to \mathbb{Z}$ $\quad \lceil x \rceil$ (round up). $\rightsquigarrow$ smallest int $\geq x$.



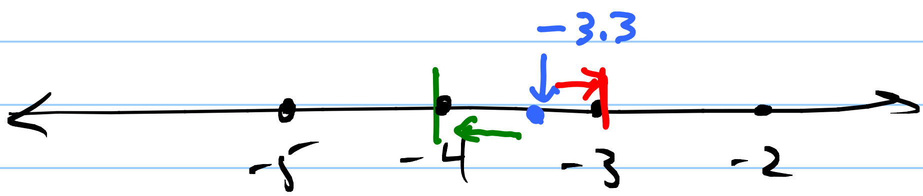
$\lfloor 2.1 \rfloor = 2$ $\qquad \lceil 2.1 \rceil = 3$

$\lfloor 2 \rfloor = 2.$ $\qquad \lceil 2 \rceil = 2.$

$\lfloor -3.1 \rfloor = -4$ $\qquad \lceil -3.1 \rceil = -3.$

Non-neg : $\lceil 3.1 \rceil = 4$ $\qquad \lceil 5 \rceil = 5$

$\lfloor 3.1 \rfloor = 3$ $\qquad \lfloor 5 \rfloor = 5$



$\lfloor -3.3 \rfloor = -4$ $\qquad \lceil -3.3 \rceil = -3.$

$\lfloor -4 \rfloor = -4 \qquad \lceil -4 \rceil = -4$

Functions can have more than one input:

$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$f(x,y) = x+y \qquad f(3,4) = 7.$

$f(x,y) = \max\{x,y\}. \qquad f(5,3) = 5.$

$f(x,y) = 2^x + 2^y$

(Find function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ one to one)

or $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$

$g(x,y) = (\max\{x,y\}, \min\{x,y\}).$

$g(3,5) = (5,3)$

$g(5,3) = (5,3)$

Inverse & Composition of functions.

Note Title 2/4/2015

Let $f : \underline{A} \to \underline{B}$

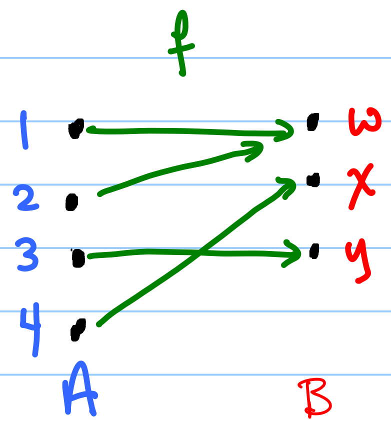
Define the inverse of f (denoted f^{-1})

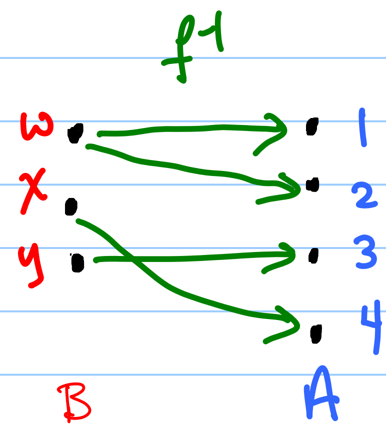
inv of # x is x^{-1}

$$f(a) = b \quad \text{iff} \quad f^{-1}(b) = a.$$

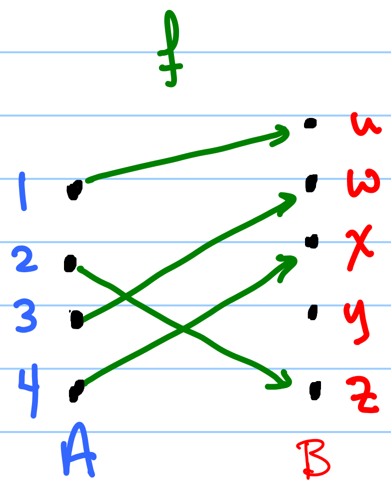
for all $a \in A$ & $b \in B$.

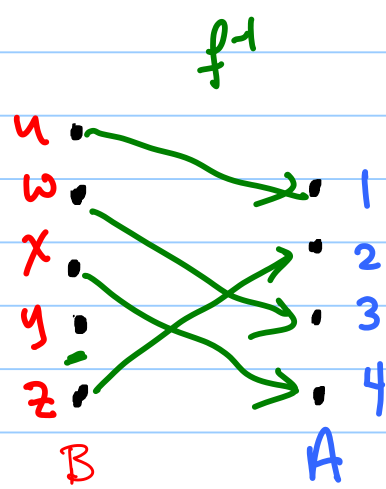
f^{-1} is not always a well defined function:



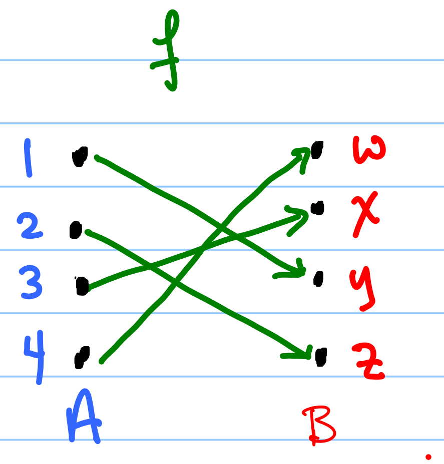


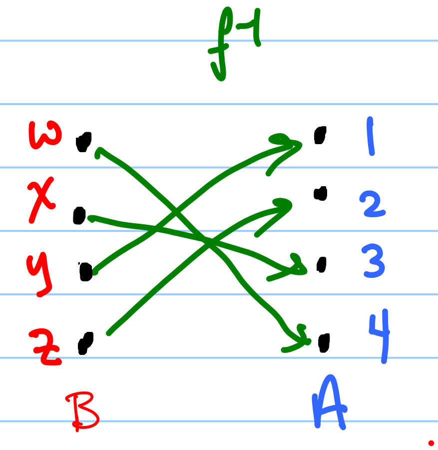
not well defined





not well defined





Theorem A function f has a well-defined inverse if and only if it is a bijection.

Examples: $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = 5x + 3$ not onto $f(\ \overset{ex}{}\ ) \overset{?}{=} 0$

$f(x) = x - 2$ $\quad f^{-1}(x) = x + 2$ $\quad f^{-1}(y) = y + 2$

$f(8) = 6$

$f(100) = 98.$ $\quad y = x - 2$ $\quad x = y + 2$

$f(x) = |x|$ $\quad f(-3) = f(3) = 3.$

$f(x) = \lfloor \frac{x}{2} \rfloor$ $\quad f(2) = \lfloor \frac{2}{2} \rfloor = \lfloor 1 \rfloor = 1.$

$f(3) = \lfloor \frac{3}{2} \rfloor = \lfloor 1.5 \rfloor = 1.$

not 1-1.