Inverse & Composition of functions.

Note Title 2/4/2015

Let $f: \underline{A} \to \underline{B}$

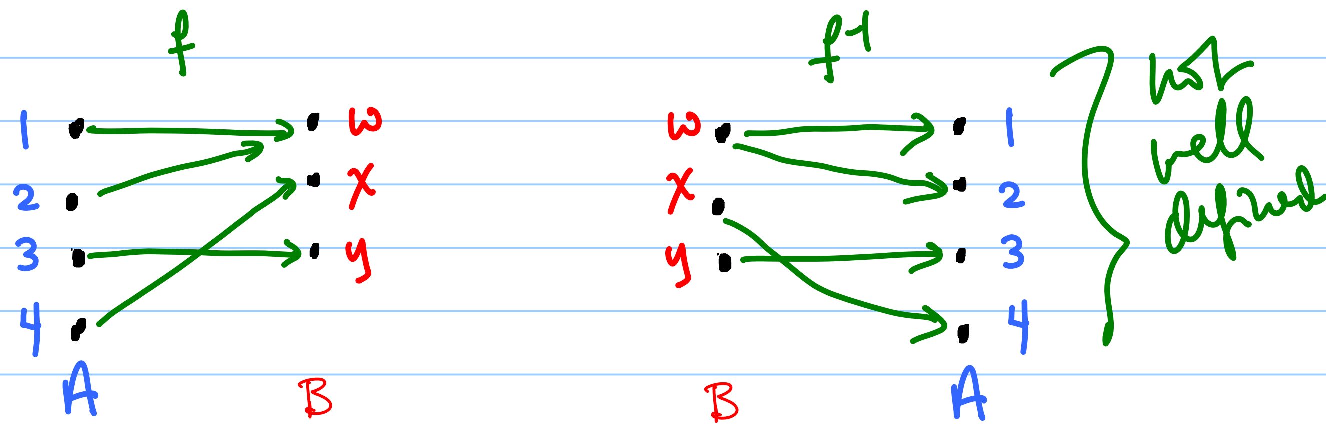
Define the inverse of f (denoted f^{-1})

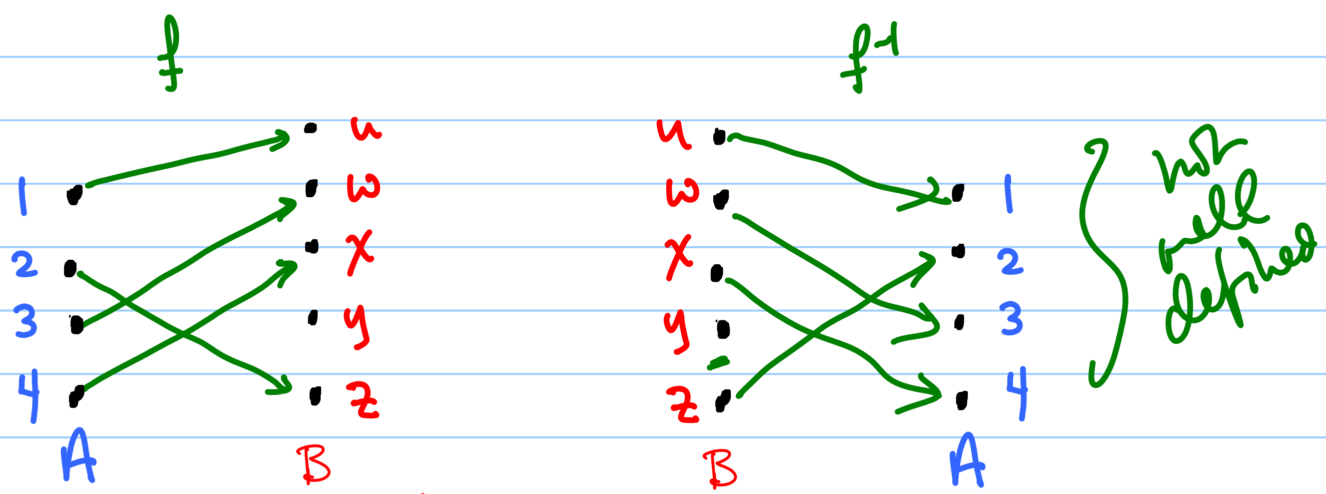
$$f(a) = b \quad \text{iff} \quad f^{-1}(b) = a.$$

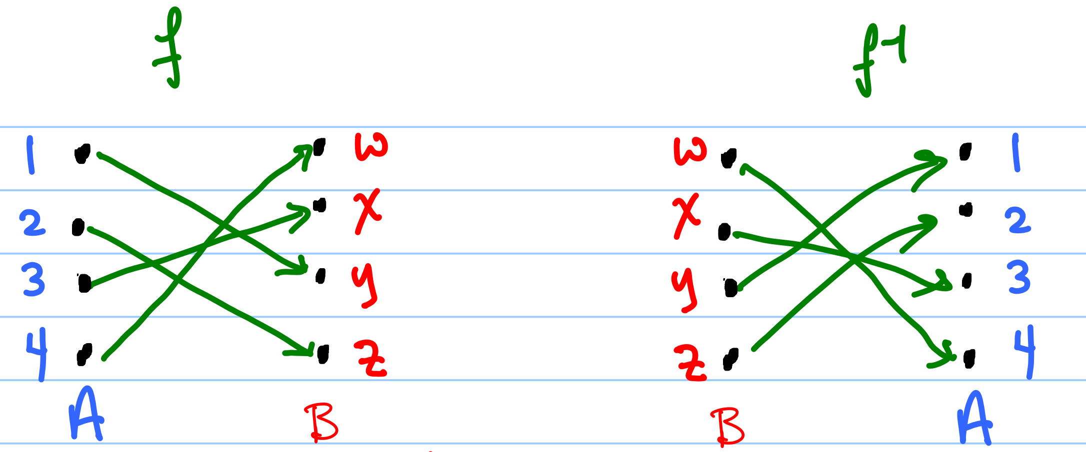
for all $a \in A$ & $b \in B$.

inv of # x is x^{-1}

f^{-1} is not always a well defined function:







Theorem A function f has a well-defined inverse if and only if it is a bijection.

Examples: $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = 5x + 3$ not onto $f(\ ) \overset{?}{=} 0$ ex

$f(x) = x - 2$ $f^{-1}(x) = x + 2$

$f(8) = 6$

$f(100) = 98$.

$y = x - 2$ $x = y + 2$ $f^{-1}(y) = y + 2$

$f(x) = |x|$ $f(-3) = f(3) = 3$.

$f(x) = \left\lfloor \frac{x}{2} \right\rfloor$ $f(2) = \left\lfloor \frac{2}{2} \right\rfloor = \lfloor 1 \rfloor = 1$.

$f(3) = \left\lfloor \frac{3}{2} \right\rfloor = \lfloor 1.5 \rfloor = 1$.

not 1-1.

One way to show that a function is a bijection is to show its inverse.

Let $A = \{1,2,3\}$.

$f : \underbrace{\{0,1\}^3}_{} \to \underbrace{P(A)}_{}$ for $i = 1,2,3$ $i \in f(x)$ if and only if the i^{th} bit of x is 1.

$f(\overset{1\,2\,3}{101}) = \{1,3\} \in P(A)$.

$f(010) = \{2\}$

$f(000) = \emptyset$

Is f a bijection?

$f^{-1} : \underbrace{P(A)}_{} \to \underbrace{\{0,1\}^3}_{}$

Let $X \subseteq A$ $\qquad f^{-1}(X) = \underline{b_1 b_2 b_3}$

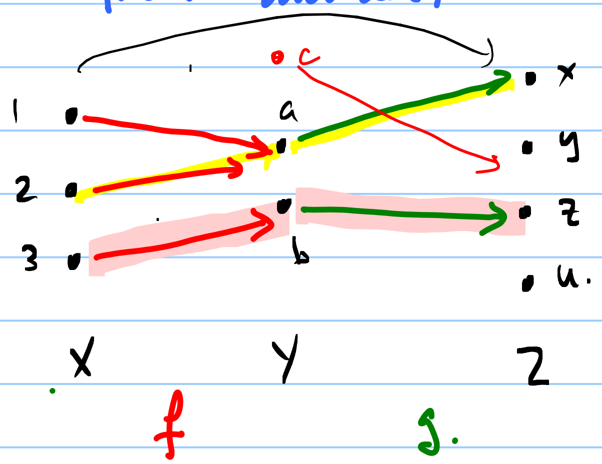
for $i = 1,2,3$
$b_i = 0$ if $i \notin X$
$= 1$ if $i \in X$.

$f^{-1}(\underbrace{\{1,2,3\}}_{}) = 111$

$f^{-1}(\{1\}) = \underset{b_1\,b_2\,b_3}{1\ 0\ 0}$

For any $X \subseteq A$ $f^{-1}(X)$ corresponds to exactly one string in $\{0,1\}^3$.

Composition of functions: applying one function & then another.



$g \circ f(2) = x$

$g \circ f(3) = z$

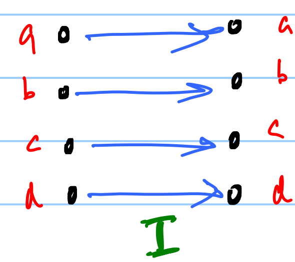
$f : X \to Y$ $\qquad$ $g : Y \to Z$

$g \circ f : X \to Z$

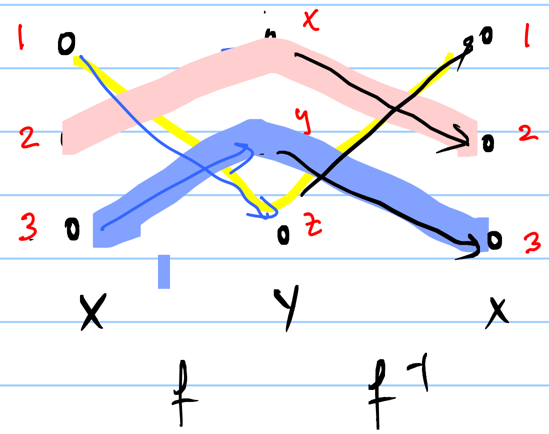
for $x \in X$ $\qquad g \circ f(x) = g(f(x))$

Identity function: $I : A \to A$ $\qquad$ Domain = Target.

For all $a \in A$ $\qquad I(a) = a$.



If f is a bijection then $f \circ f^{-1} = f^{-1} \circ f = I$



$f(x) = x^2$ $\quad f: \mathbb{R} \to \mathbb{R}$

$g(x) = \left\lfloor \frac{x}{3} \right\rfloor$ $\quad g: \mathbb{R} \to \mathbb{R}$.

$f \circ g(5) = f(g(5)) = f(1) = 1$

$g \circ f(5) = g(f(5)) = g(25) = 8$

Boolean Algebra

Note Title 2/6/2015

Rules & operations in working with variables are 0 or 1.

Mathematically equivalent to logic.
↳ Sometimes called "Boolean logic".

$0 \equiv F$

$1 \equiv T$

Multiplication:

$0 \cdot 0 = 0$

$0 \cdot 1 = 0$

$1 \cdot 0 = 0$

$1 \cdot 1 = 1$

Addition

$0 + 0 = 0$

$0 + 1 = 1$

$1 + 0 = 1$

$1 + 1 = 1$

Complement

$\overline{0} = 1$

$\overline{1} = 0$

Boolean variables have a value of 0 or 1.

Rules of precedence: mult before addition. → parens can override.

complement applied to entire value under the bar:

$x + (y \cdot z)$

$(x + y) \cdot z$

$\overline{0+1} = \overline{1} = 0.$

$0 + \overline{1} = 0 + 0 = 0.$

$$0 + (\overline{0 \cdot 1}) = 0 + (\overline{0}) = 0 + 1 = 1.$$

$$x + (y \cdot \overline{z})$$

$x = 0, \quad y = 0, \quad z = 1.$

$$x + (\overline{y \cdot z})$$

$$0 + (0 \cdot \overline{1}) =$$

$$0 + 0 = 0.$$

Abbreviation: $x \cdot y \Rightarrow xy$

$1 \cdot x \quad \cancel{1x}$

All of the usual laws of logic apply to Boolean variables:

De Morgan: $\overline{x + y} \equiv \overline{x}\,\overline{y}$

$$\neg(x \vee y) \equiv \neg x \wedge \neg y$$

holds regardless of whether x & y are 0 or 1.

General Boolean Algebra:

Any mathematical "system" (i.e set) that has:

- Element representing 1.
- Element representing 0.
- $+$ $\cdot$ complement operators.
- Obeys: assoc
  comm
  distr
  Identity: $x+0=x$ $\quad x\cdot 1=x$
  Compl: $x+\bar{x}=1$ $\quad x\bar{x}=0$

For any finite set A $\quad$ P(A). $\rightarrow$ elements.

addition $\longleftrightarrow$ Union
multiplication $\longleftrightarrow$ intersection.
"1" $\longleftrightarrow$ A $\cdot$
"0" $\longleftrightarrow$ $\emptyset$ $\cdot$
Complement $\longleftrightarrow$ complement.

$|P(A)| = 8$

$\{3\} \in P(A)$

$A = \{1, 2, 3\}.$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Complement: $x + \bar{x} = 1$

$x \cdot \bar{x} = 0$

Suppose $x = \{1,3\}.$

$\bar{x} = \{2\}.$

$1 = \{1,2,3\}$

$0 = \phi$

$\{1,3\} \cup \{2\} = \{1,2,3\} = A$

$\{1,3\} \cap \{2\} = \phi$

Boolean Functions

$$f: \{0,1\}^n \to \{0,1\}$$

Maps one or more boolean variables to 0/1.

Can specify a boolean function by an output table

x	y	$f(x,y)$
0	0	0
0	1	1
1	0	1
1	1	0

Can also specify a boolean function using Boolean operations. Boolean "expression"

$$f(x) = x\bar{y} + \bar{x}y$$

$x = 0$
$y = 1$

$$= \underbrace{0 \cdot \bar{1}}_{0 \cdot 0} + \bar{0} \cdot 1$$

$$0 \cdot 0 + 1 \cdot 1 = 0 + 1 = 1 \checkmark$$

A Boolean expression always has an equivalent input/output table.

Is the converse always true?

A *literal* for variable x is x or $\bar{x}$

If f is a boolean function with input variables $x_1, \dots, x_k$

a $\boxed{\text{minterm}}$ for f is a product of k literals in which each input variable or its negation (but not both) appears exactly once.

$x_1 \cdot \bar{x}_2 \cdot \bar{x}_3 \cdot x_4 \cdots \cdot \bar{x}_k$

$f(x,y,z)$

	minterms:	not minterms:
	$x\bar{y}\bar{z}$	$x\bar{y}\bar{z}x$
	$\bar{y}\bar{x}z$	$x\bar{y}\bar{z}\bar{x}$
		$y\bar{z}$

Example:

x	y	z	$f(x,y,z)$	
0	0	0	0	
0	0	1	1	$\longrightarrow$ $\bar{x}\bar{y}z$
0	1	0	0	
0	1	1	1	$\longrightarrow$ $\bar{x}yz$
1	0	0	0	
1	0	1	0	
1	1	0	0	
1	1	1	1	$\longrightarrow$ xyz

$$f(x,y,z) = \bar{x}\bar{y}z + \bar{x}yz + xyz$$

$$z(\bar{x}\bar{y} + \bar{x}y + xy)$$

$$z(\bar{x}(\bar{y}+y) + xy)$$

$$z(\bar{x} \cdot 1 + xy)$$

$$z(\bar{x} + xy)$$