Note Title 2/10/2015

Arbitrary boolean function (expressed by input/output table) $\rightsquigarrow$ Equivalent Boolean expression using only addition, multiplication, complement operations.

The set of operations: {addition, multiplication, complement} is functionally complete.

Can we remove one of the operations + still remain functionally complete.

De Morgan: $\overline{xy} = \bar{x} + \bar{y}$

$\Rightarrow xy = \overline{\overline{xy}} = \overline{\bar{x}+\bar{y}}$ } Computes the product without the multiplication op!

$\overline{\overline{xy}}$

Apply within more complex expressions to eliminate all multiplication operations.

$\overline{(x+y)}\,\bar{z} = \overline{x+y+z}$

$(xy+z)w = \overline{\overline{(\overline{\bar{x}+\bar{y}}+z)}\,\bar{w}}$

$= \overline{\overline{\overline{\bar{x}+\bar{y}}+z}+\bar{w}}$

The set $\{$addition, complement$\}$ is functionally complete.

Arbitrary boolean function (expressed by input/output table) $\longrightarrow$ Boolean expression using $+$, $\cdot$, $-$ operations.

$\Big\Downarrow$ Remove multiplication

Boolean expression using only addition + complement.

What about the set $\{$multiplication, complement$\}$?

De Morgan again!

$$\overline{x+y} = \bar{x}\,\bar{y}$$

$$x+y = \overline{\overline{x+y}} = \overline{\bar{x}\,\bar{y}}$$

Can use this rule to eliminate any addition operation in a Boolean expression.

$$(x+y)z \equiv \left(\overline{\bar{x}\,\bar{y}}\right)z$$

$$xy + z \longrightarrow \overline{\overline{xy}\;\bar{z}}$$

$$xy + z(x+y) = xy + z\left(\overline{\bar{x}\,\bar{y}}\right)$$

$$\equiv \overline{\overline{xy} \cdot \overline{z(\overline{xy})}}$$

The set {multiplication, complement} is functionally complete.

Arbitrary boolean function (expressed by input/output table) $\longrightarrow$ Boolean expression using $+$, $\cdot$, $-$ operations.

Remove addition:

Boolean expression using only multiplication + complement.

Is the set {addition, multiplication} functionally complete?

No. Can't compute $\overline{x}$.

Is there a single operation which is functionally complete all by itself?

Yes!

NAND $= x \uparrow y \equiv \overline{xy}$

x	y	$x \uparrow y$
0	0	1
0	1	1
1	0	1
1	1	0

Only need to show how to compute $\overline{x}$ and xy with $\uparrow$.

I/O table $\Longrightarrow$ Boolean expression w/ $+$, $\cdot$, $\overline{}$ $\longrightarrow$ Boolean expression w/ $\cdot$, $\overline{}$

replace $+$ using $\cdot$ $\overline{}$

$\Downarrow$

express $\cdot$ using $\uparrow$

express $\overline{}$ using $\uparrow$

Boolean expression w/ $\uparrow$.

x	$x \uparrow x$
0	$0 \uparrow 0 = 1$
1	$1 \uparrow 1 = 0$

$$\overline{a} = a \uparrow a$$

$$xy = \underline{\overline{x \uparrow y}} = (x \uparrow y) \uparrow (x \uparrow y)$$

$$\overline{xy} = x \uparrow y \qquad (a = x \uparrow y)$$

$$xy = \overline{x \uparrow y} = (x \uparrow y) \uparrow (x \uparrow y)$$

x	y	$x \uparrow y$	$(x \uparrow y) \uparrow (x \uparrow y) = xy$
0	0	1	$1 \uparrow 1 = 0$
0	1	1	$1 \uparrow 1 = 0$
1	0	1	$1 \uparrow 1 = 0$
1	1	0	$0 \uparrow 0 = 1$

$\overline{\bar{x}\, y} = \bar{x} \uparrow y$ $\qquad\qquad$ $\underbrace{\overline{\bar{x}\, y}} = \underbrace{\bar{x} \uparrow y}$

$\bar{x}y + \bar{z} = \overline{\overline{\bar{x}\, y}\ \bar{\bar{z}}} = \overline{\overline{\bar{x}\, y}\ z}$

$\overline{\overline{\bar{x} y}\ z}$

$\overline{(\bar{x} \uparrow y)\, z}$

$\overline{((\bar{x} \uparrow y) \uparrow z) \uparrow ((\bar{x} \uparrow y) \uparrow z)}$

$(((\underbrace{\bar{x} \uparrow y}) \uparrow z) \uparrow ((\bar{x} \uparrow y) \uparrow z)) \uparrow (((\bar{x} \uparrow y) \uparrow z) \uparrow ((\bar{x} \uparrow y) \uparrow z))$

$((x \uparrow x) \uparrow y)$

Note Title — Boolean Satisfiability — 2/11/2015

Boolean Satisfiability

SAT Input: Boolean expression with n input variables.

Output: Yes/No : Is there a setting of the input variables that cause the boolean expression to evaluate to 1?

$$f(x,y,z) = z(x+\bar{y})(\bar{z}+\bar{x})$$

$= 1.$

$z=1$, $x=0$, $y=0$

$$f(x,y,z) = z(x+\bar{y})(\bar{z}+\bar{x})(y+\bar{z})$$

Not Satisfiable.

$z=1$, $y=1$, $x=0$

$$f(x,y,z) = (z+x+\bar{y})(y+\bar{x}+z)(\bar{z}+x)(\bar{x}+y+\bar{z})$$

One way to check is by truth table:

$$f(x,y,z) = \underline{(z + x + \overset{1}{\bar{y}})}\,\underline{(y + \overset{1}{\bar{x}} + z)}\,\underline{(\overset{1}{\bar{z}} + x)}\,(\overset{1}{\bar{x}} + y + \bar{z})$$

x	y	z	f(x,y,z)
0	0	0	1
0	0	1	
0	1	0	
0	1	1	
1	0	0	
→ 1	0	1	0
1	1	0	
1	1	1	

If a Boolean expression has 200 variables, this will require the function to be evaluated as many as 2^{200} times.

1 nanosecond per evaluation:
... over a billion years!

Why is SAT important?

The language of Boolean logic can be used to express many important & practical computational problems.

Efficient algorithm for SAT ⇒ Efficient algorithm for all those problems.

Scheduling Classes.

Set of classes.
Set of time slots.

Certain pairs of classes can not be assigned to the same time slot.

Is there a way to schedule the classes that does not violate any of the constraints?

I : input to scheduling problem.
n : # of classes.
k : # of time slots.
Pairs (i,j) class j and i can not be scheduled during the same time.

$\Longrightarrow$

Boolean expression E.
E is satisfiable if + only if there is a feasible schedule for I.

For example: 5 classes $\{A, B, C, D, E\}$.
2 time slots
(A,B) (C,D) (D,E) (B,E) (A,D).

Boolean variables: $x_{A1}, x_{B1}, x_{C1}, x_{D1}, x_{E1}$
$x_{A2}, x_{B2}, x_{C2}, x_{D2}, x_{E2}$.

$x_{C2} = 1$ if class C is scheduled during time slot 2.

$$\underbrace{(x_{D1} + x_{D2})}_{\text{D must be scheduled in at least one of the two time slots}} \underbrace{\overset{\overline{x_{D1} \cdot x_{D2}}}{(\overline{x}_{D1} + \overline{x}_{D2})}}_{\text{D cannot be scheduled in both time slots.}}$$

Constraint (D,E) D & E can not both be scheduled in the same time slot:

$$\underbrace{(\overline{x}_{D1} + \overline{x}_{E1})}_{\text{D \& E can not both be scheduled in time slot 1}} \underbrace{(\overline{x}_{D2} + \overline{x}_{E2})}_{\text{D \& E can not both be scheduled in time slot 2.}}$$

Put together in one big product:

$$(X_{A1} + X_{A2})(\overline{X}_{A1} + \overline{X}_{A2})(X_{B1} + X_{B2})(\overline{X}_{B1} + \overline{X}_{B2})$$

$$(X_{C1} + X_{C2})(\overline{X}_{C1} + \overline{X}_{C2})(X_{D1} + X_{D2})(\overline{X}_{D1} + \overline{X}_{D2})$$

$$(X_{E1} + X_{E2})(\overline{X}_{E1} + \overline{X}_{E2})$$

times:

$$(\overline{X}_{A1} + \overline{X}_{B1})(\overline{X}_{A2} + \overline{X}_{B2})(\overline{X}_{C1} + \overline{X}_{D1})(\overline{X}_{C2} + \overline{X}_{D2})$$

$$(\overline{X}_{D1} + \overline{X}_{E1})(\overline{X}_{D2} + \overline{X}_{E2})(\overline{X}_{B1} + \overline{X}_{E1})(\overline{X}_{B2} + \overline{X}_{E2})$$

$$(\overline{X}_{A1} + \overline{X}_{D1})(\overline{X}_{A2} + \overline{X}_{D2})$$