Note Title 2/18/2015

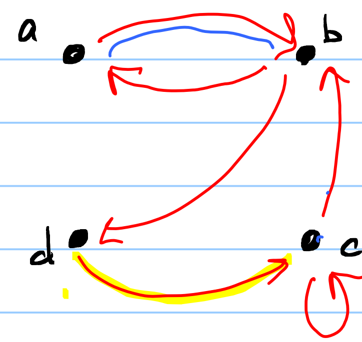
Directed Graphs

Relation R on A
$R \subseteq A \times A$

$G = (V, E)$ V = set of vertices.
E = set of edges. subset of $V \times V$.

$V = \{a, b, c, d\}$

$E = \{(a,b), (b,a), (b,d), (c,b), (d,c), (c,c)\}$



Tail of edge (a,b) is a

Head of edge (a,b) is b.

Indegree of c : 2

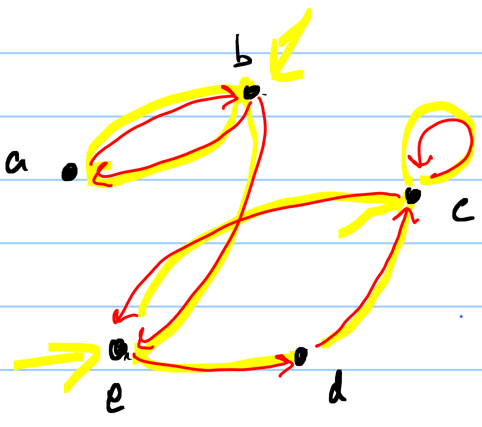
Outdegree of c : 2.

Indegree of v = $|\{u \mid (u,v) \in E\}|$

Out-degree = $|\{u \mid (v,u) \in E\}|$

A <u>walk</u> in a graph is an alternating sequence of vertices & edges starting & ending with a vertex.
Each edge comes after its tail & before its head:

$$\dots\ u,\ (u,v),\ v,\ \dots\dots$$



$$\langle b, (b,a), a, (a,b), b, (b,e), e, (e,d), d, (d,c), c, (c,c), c, (c,e), e \rangle$$

Including the edges is redundant:

$$\langle b, a, b, e, d, c, c, e \rangle$$

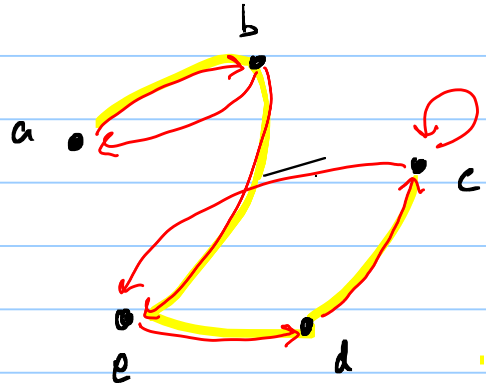
length of the path is the # of edges (7).

$\langle b, a, e \rangle$ is <u>not</u> a walk because

(a,e) is not an edge.

Walk : $\langle v_0, v_1, v_2, \ldots, v_\ell \rangle$

A walk is also a path if it has no repeated vertices $\forall i,j \quad v_i \neq v_j$.

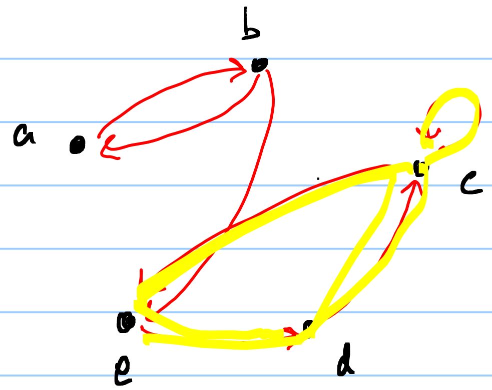


$\langle a, b, e, d, c \rangle$ path.

$\langle e, d, c, e \rangle$ no

$\langle d, c, c, e \rangle$ no.

A walk is a circuit if start & end vertices are the same. ($v_0 = v_\ell$).



$\langle e, d, c, e \rangle$ Yes.

$\langle a, b, a \rangle$ Yes.

$\langle e, d, c, c, e, d, c, e \rangle$

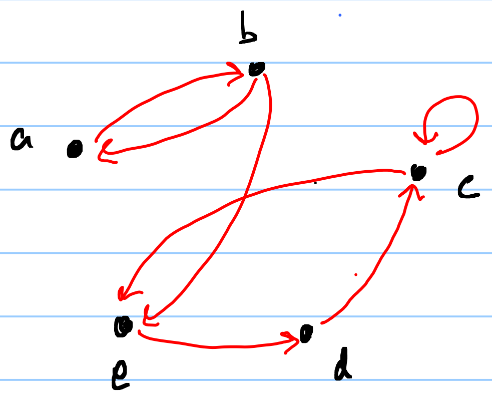
Yes.

A circuit is a cycle if there are no repeated vertices except the first and the last vertex.

$v_0 = v_\ell$

if $i, j \in \{1, 2, \ldots, \ell\}$

$v_i \neq v_j$.



$\langle a, b, a \rangle$ Yes

$\langle e, d, c, e, d, c, e \rangle$ No.

$\langle e, d, c, e \rangle$ Yes.

$\langle c, c, e, d, c \rangle$ No.

Transitive Closure :

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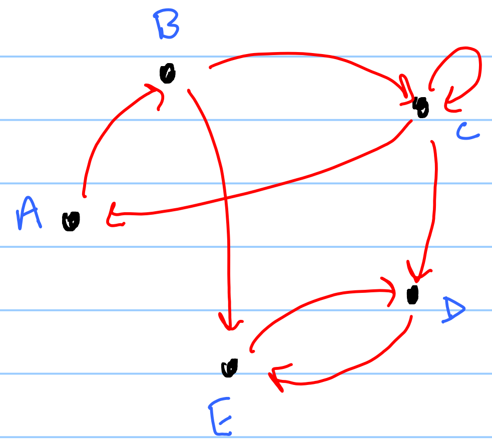
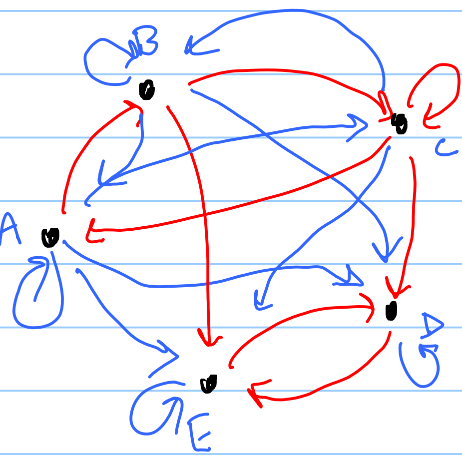


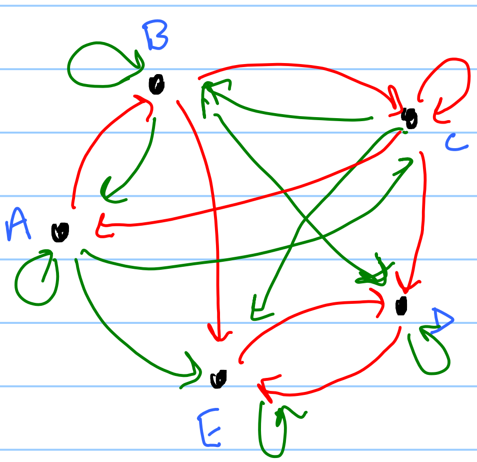
Is it possible to travel from vertex C to Vertex E by a walk?

From vertex E to C?



Build a new graph. Same vertex set as G.

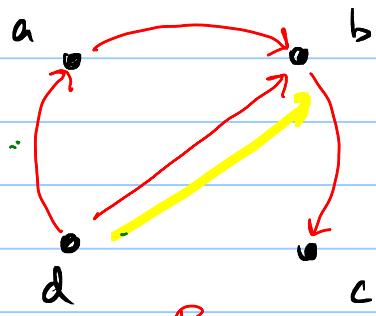
Edge (X,Y) whenever Y can be reached from X by a walk.



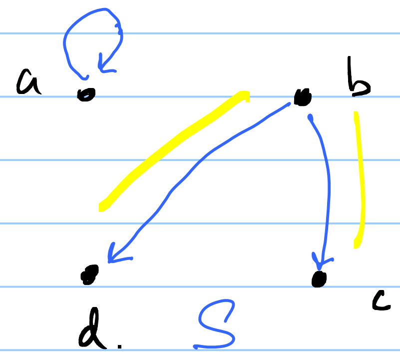
What is the smallest set of edges we need to add to G in order to make it transitive?

Transitive Closure.

Composition of two relations on the same set.

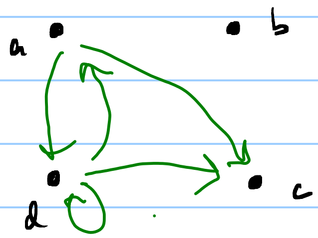


R

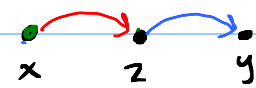


S

$S \circ R$

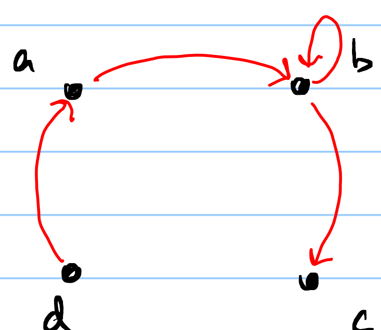


add an edge from x to y if there is a red-blue walk:

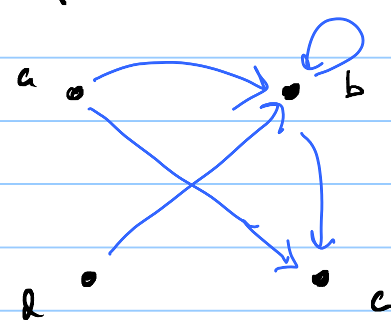


$\boxed{\text{Can have } x=z \text{ or } x=y \text{ or } z=y.}$

Compose a relation R with itself.

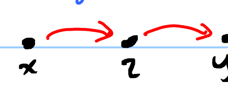


R



$R \circ R$

Edge from x to y if there is a walk of length 2 from x to y in R.



Can compose a relation multiple times with itself:

$$R^1 = R$$

$$R^2 = R \circ R$$

$$R^3 = R \circ R \circ R = R^2 \circ R$$

$$\cdots R^k = \underbrace{R \circ R \circ \cdots \circ R}_{k \text{ times.}} = R^{k-1} \circ R$$

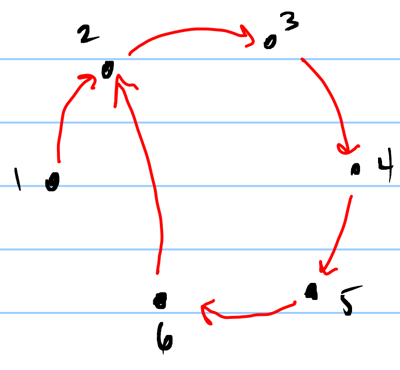
$\Rightarrow$ k^{th} power of R.

Directed graph G:

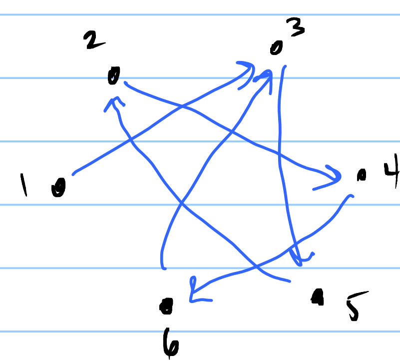
G^k has the same vertex set as G.

(x,y) is an edge in G^k iff there is a walk of length k in G.

Theorem: The edge E defines a relation on V. The edge set of G^k is $E^k = \underbrace{E \circ E \circ \cdots \circ E}_{k \text{ times.}}$



G

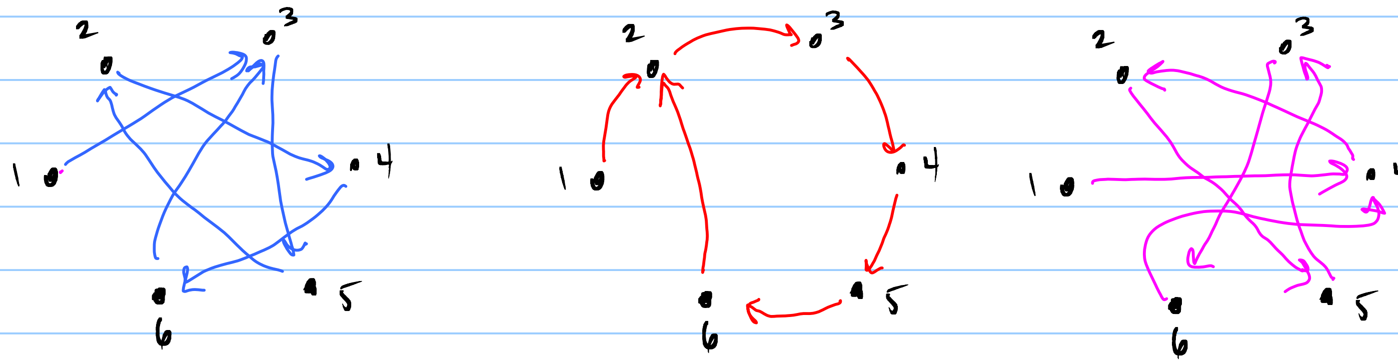


G^2 find $E \circ E$.

$R \circ S \neq S \circ R$

$A^k \circ A = A \circ A^k$

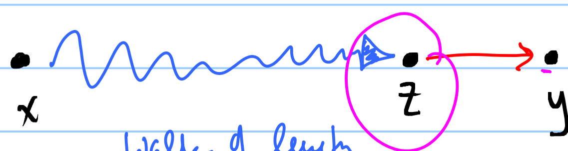
$G \circ G^2$



G^2 $\quad$ G $\quad$ G^3

Is there a walk from x to y of length k?

Iff there is a z (possibly = to x or y)



walk of length $k-1$ from x to z

edge from z to y in G.

(x,z) is an edge in G^{k-1}

$G^{k-1} \circ G$

$G^+ = G^1 \cup G^2 \cup G^3 \cup \ldots\ldots$

Same vertex set. Take union of the edges.

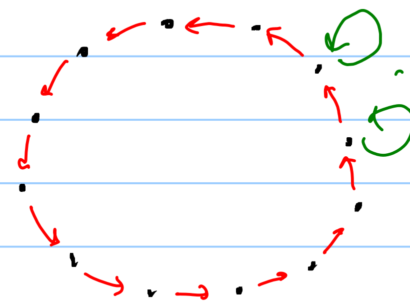
(x,y) is an edge in G^+ if there is a walk from x to y in G of any length.

Don't need an infinite union:

Fact: If there is a walk from x to y in G, then there is a walk from x to y in G of length $\leq n$. $\Rightarrow n = |V|$.

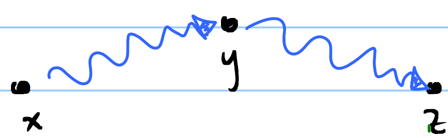
$$\boxed{G^+ = G^1 \cup G^2 \cup \cdots \cup G^n}$$

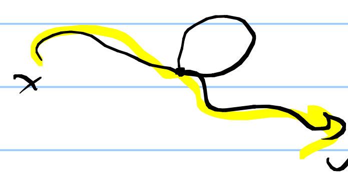
Worst case:



The edge set of G^+ is transitive:

Walk from x to y + Walk from y to z $\Rightarrow$ Walk from x to z.





Same for relation R.

$$R^+ = R^1 \cup R^2 \cup \cdots \cup R^n$$

R^+ is the transitive closure of R: the smallest relation that includes all of R and is transitive.

Another way to find the transitive closure of R:

Repeat until no new pairs are added:

$$\text{If } (x,y) \in R \wedge (y,z) \in R \wedge (x,z) \notin R$$

$$\text{add } (x,z) \text{ to } R.$$