Note Title 2/23/2015

Relation R on a set A

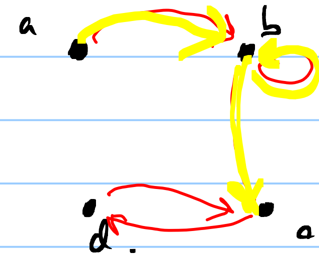
Directed graph $G = (V, E)$

$A \longleftrightarrow V$

$R \longleftrightarrow E$.

$A = \{a, b, c, d\}$

$R = \{ (a,b), (b,b), (b,c), (c,d), (d,c) \}$



Graph powers of a directed graph $G = (V, E)$

$G^k = (V, E^k)$

relation E composed with itself k times!

$E^k = E \circ E \circ E \circ \cdots \circ E$

$(x,y) \in E^k$ iff there is a walk from x to y in G of length exactly k.

$E^1 = E$

$E^2 = E \circ E$

$E^3 = E^2 \circ E = E \circ E^2$

$E^4 = E^3 \circ E = E^3 \circ E$

$\vdots$

$G^+ = (V, E^+)$

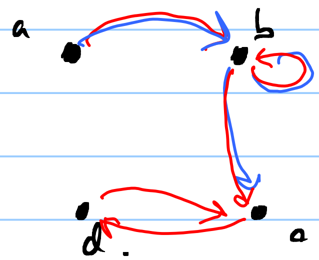
(x,y) is an edge in G^+ iff there is a walk of any length from x to y in G.

$G^+ = G^1 \cup G^2 \cup \cdots$

$= G^1 \cup G^2 \cup \cdots \cup G^n$.

Same vertex set, take the union of all the edges.

Adjacency Matrix for Graph G:



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

An $n \times m$ matrix over a set S:

A_{ij} row col

$$\begin{array}{cc} & \text{column } j \\ & m \\ A = n & \begin{bmatrix} A_{11} & \cdots & \cdots & A_{1m} \\ \vdots & & A_{ij} & \vdots \\ A_{n1} & \cdots & \cdots & A_{nm} \end{bmatrix} \text{ row } i \end{array}$$

All the $A_{ij} \in S$.

The adjacency matrix for a graph G with n vertices is an $n \times n$ matrix over $\{0, 1\}$.

$$\begin{bmatrix} 3.2 & -4.1 & 6.7 & 1.5 \\ 1.5 & -2 & 100 & 2.3 \\ 5 & 7 & -51 & 7.1 \end{bmatrix}$$

a 3×4 matrix over $\mathbb{R}$.

If addition and multiplication is defined for the set S, then can multiply and add some pairs of matrices over S.

↳ We will use Boolean multiplication and addition on square matrices.

rows = # columns.

Can add matrices A & B if A & B are both $n \times m$ matrices.
(i.e. A & B have the same # rows and same # columns).

Can compute $A \cdot B$ if # columns of A = # rows of B.

If A & B are square matrices, can compute

$A + B$ and $A \cdot B$ or $B \cdot A$

if A & B have same # rows & columns.

$$
A = \begin{array}{r l}
 & \begin{bmatrix} & & & \\ A_{i1} & A_{i2} & \cdots & A_{in} \end{bmatrix} \\
\end{array}
$$

row i $\rightarrow$ $A_{i1} \; A_{i2} \; \cdots \; A_{in}$

$$
B = \begin{array}{c} \text{column } j \\ \begin{bmatrix} B_{j1} & & \\ B_{j2} & & \\ \vdots & & \\ B_{jn} & & \end{bmatrix} \end{array}
$$

dot product of row i from A and col j from B is

$$(A_{i1} \cdot B_{1j}) + (A_{i2} \cdot B_{2j}) + \cdots + (A_{in} \cdot B_{nj}) =$$

$$
A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad
B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}
$$

Row 2 of A
Col 3 of B. $= 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 1\cdot 1 = 0+1+0+1 = 1$

Row 3 of A
Col 2 of B $\quad 0\cdot 1 + 0\cdot 1 + 1\cdot 0 + 1\cdot 0 = 0$

The product of two $n \times n$ matrices A & B

$(AB)_{ij}$ = dot product of row i of A and column j of B.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 1 & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Matrix multiplication is associative:

$$(AB)C = A(BC)$$

not commutative: $\underline{AB \neq BA}$ (in general).

Powers of a matrix A:

$A^1 = A$

$A^2 = A \cdot A$

$A^3 = (A \cdot A) A = A^2 \cdot A = A \cdot A^2$

$A \cdot (A \cdot A) = A \cdot A^2$

$A^4 = A \cdot A \cdot A \cdot A = A^3 \cdot A = A \cdot A^3$

$A \cdot A^2 = A^2 \cdot A$

$A^4 = A^3 \cdot A = A \cdot A^3$

$= A^2 \cdot A^2$