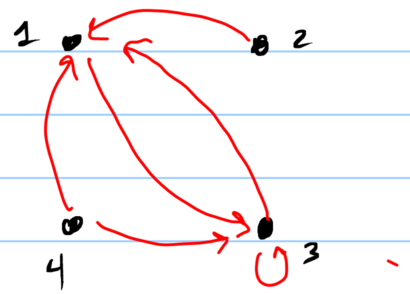
Note Title 2/25/2015



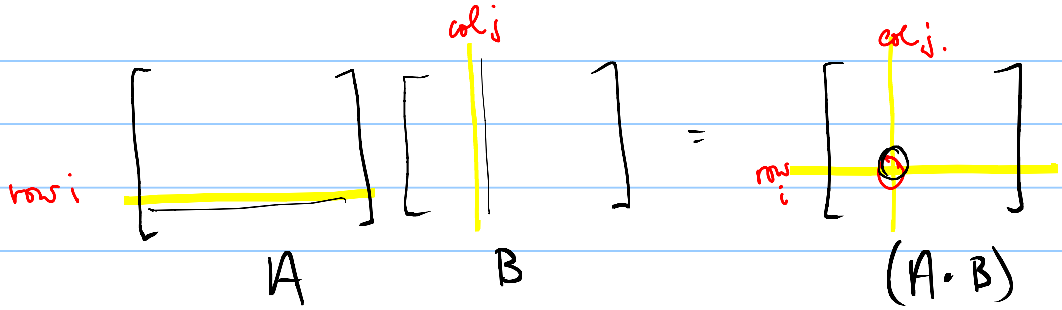
Directed graph G.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

Adjacency matrix for G

Defined matrix multiplication for Boolean square matrices.

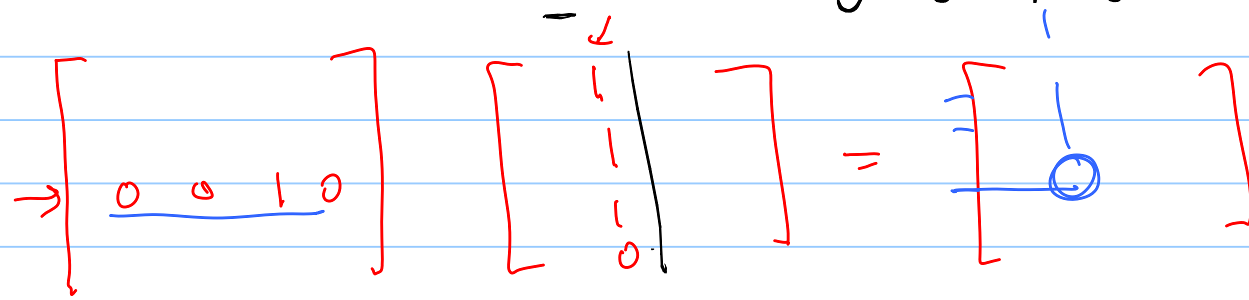
$(AB)_{ij}$ = dot product of row i from A and column j from B.



Using Boolean multiplication & addition:

i.e.: $1 + 1 = 1$.

$$\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline \end{array}$$
$0 + 0 + 1 + 0 = 1$



Directed graph G with adjacency matrix A.

$G^k = (V, E^k)$

E^k: (x,y) is an edge iff there is a path of length k from x to y in G.

A^k is the adjacency matrix for G^k.

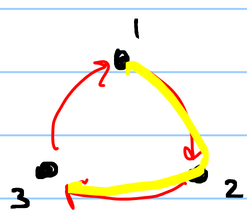
A^k: matrix multiplication

G^k → graph on V representing paths of length k in G.

$$\begin{tikzcd} & G^k \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ A^k \arrow[rr, leftrightarrow] & & E^k \end{tikzcd}$$

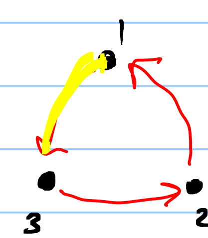
A^k → matrix A multiplied times itself k times.
(Boolean mult & add).

E^k ↳ relation E composed with itself k times.
$E \circ E \circ \cdots \circ E$



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array} = A.$$

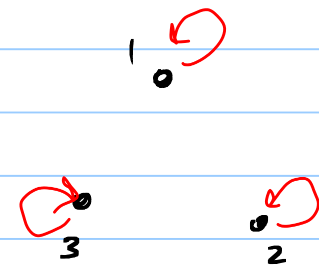
$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



G^2

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

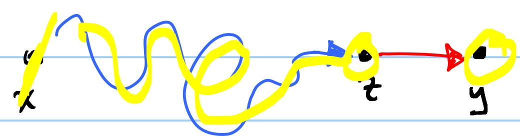
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



G^3.

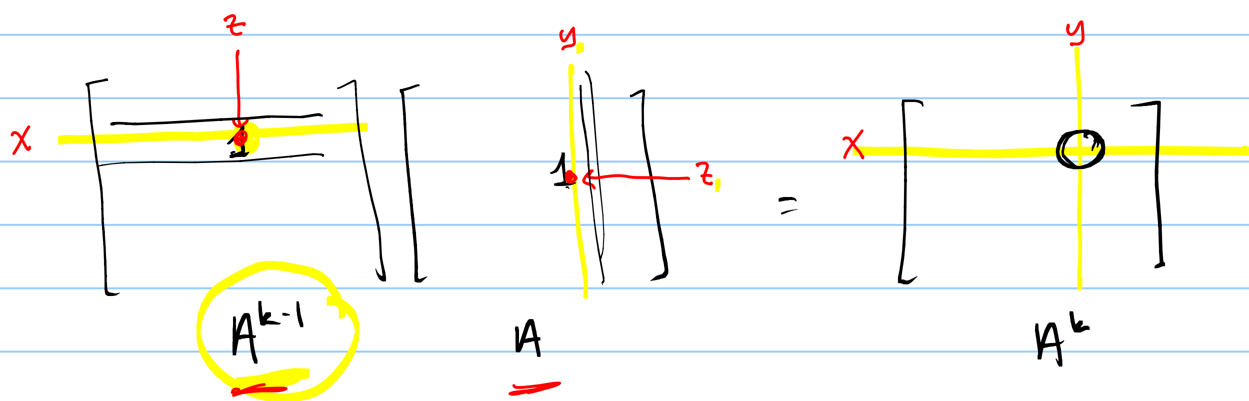
Why?? Walk of length k from x to y:

There is a z such that:



Walk of length k−1 + 1 extra edge.

$(A^{k-1})_{xz} = 1$ and $(A)_{zy} = 1$.



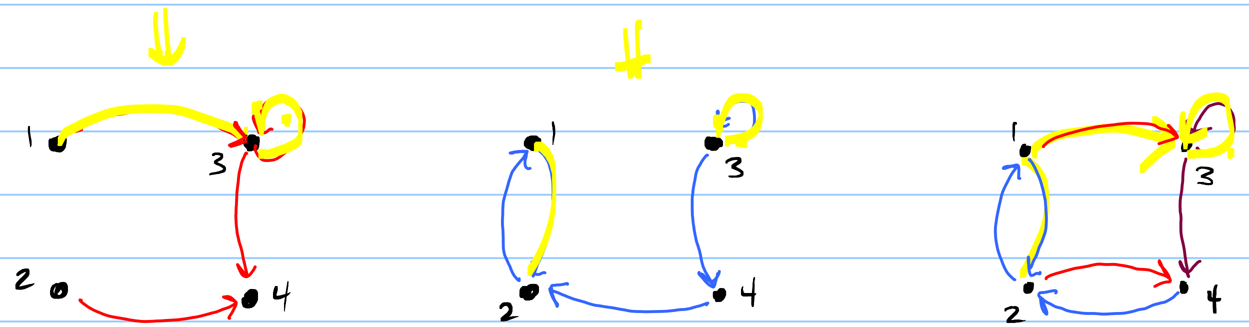
$$(A^{k-1})_{x1} A_{1y} + (A^{k-1})_{x2} A_{2y} + \dots + (A^{k-1})_{xz} A_{zy} + \dots + (A^{k-1})_{xn} A_{ny}$$

What about $\underline{G^+}$?

$$\underline{G^+} = G^1 \cup G^2 \cup G^3 \cup \cdots \cup G^n.$$

Union by matrix addition.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} = \begin{bmatrix} (A_{11}+B_{11}) & (A_{12}+A_{12}) & (A_{13}+B_{13}) \\ (A_{21}+B_{21}) & (A_{22}+B_{22}) & (A_{23}+B_{23}) \end{bmatrix}$$



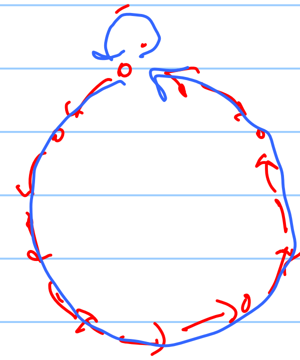
G   H $G \cup H$

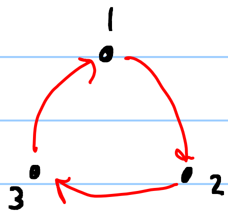
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Digraph G with adjacency matrix A.

Adjacency matrix for G^+ is:

$A + A^2 + \dots + A^n$.





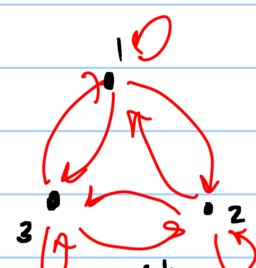
G

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

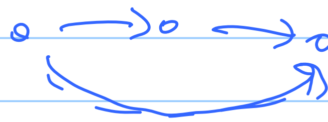
$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

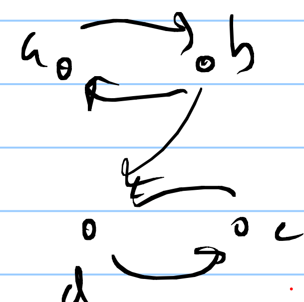
$$A^+ = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



G^+

Transitive closure





$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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Partial Order
Strict Order
Equivalence Relations

} Relations with a specific set of properties.

Partial Order: Relation R on a set A that is:

- $\Rightarrow$ Reflexive
- $\Rightarrow$ Anti-symmetric
- $\Rightarrow$ Transitive.

Notation: $x, y \in R$ $\quad xRy$ is denoted $x \preceq y$.

$x \preceq y$

Standard example: $(\mathbb{Z}, \leq)$

set $\quad$ relation.

x is "related to" y if $x \leq y$.

- Reflexive: $x \leq x$ ✓
- Anti-symmetric: $x \leq y$ and $y \leq x \Rightarrow x = y$.
- Transitive: $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

Example: $(\mathbb{Z}^+, \preceq_P)$ $\quad x \preceq_P y$ iff $\exists\, n \in \mathbb{N}$ $x^n = y$.

- Reflexive: $x \preceq_P x$? $\quad x^1 = x$
- Anti-symmetric: $x \preceq_P y \wedge y \preceq_P x \Rightarrow x = y$ $\quad 3 \preceq_P 9$
- Transitive:

$y = x^n \quad z = y^m \qquad z = x^{?}\ n \cdot m.$

$3^2 = 9$

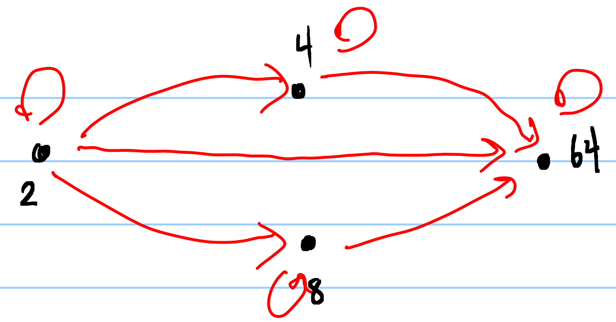
$3 \preceq_P 27$

$x^n = y \qquad x \preceq_P y \qquad x^n = y \qquad 3^3 = 27$

$y^m = x \qquad y \preceq_P x$

$9 \overset{?}{\not\preceq} 27$

$z = y^m = (x^n)^m = x^{n \cdot m}$



$2^2 = 4$

Is $4 \preceq_P 8$
or $8 \preceq_P 4$
? No $\Rightarrow$ 4 & 8 are incomparable.

4 and 8 are <u>incomparable</u>

Two elements x & y in a partial order are incomparable if $x \not\preceq y$ and $y \not\preceq x$

What about $(\mathbb{Z}, \leq)$
are any two elements incomparable? Yes

A partial order $(A, \preceq)$ is also a total order if $\forall$ $x, y \in A$ $x \preceq y$ or $y \preceq x$.

inclusive or.

Example $(\mathbb{Z} \times \mathbb{Z}, \preceq)$

$(x,y) \preceq (a,b)$ if $x \le a$ and $y \le b$

Partial Order?

Reflexive? $(x,x) \preceq (x,x)$? Yes.

Anti-Symmetric? Yes.

$(x,y) \preceq (a,b)$ $x \le a$ $y \le b$

$(a,b) \preceq (x,y)$ $\underbrace{a \le x}_{x=a}$, $\underbrace{b \le y}_{b=y}$.

Transitive?

$(x,y) \preceq (a,b) \preceq (c,d)$ Yes!

Total order? NO.

$(2,5) \not\preceq (5,2)$

$\not\succeq$

Incomparable