Note Title 2/25/2015

Partial Order
Strict Order
Equivalence Relations

} Relations with a specific set of properties.

Partial Order: Relation R on a set A that is:

- Reflexive
- Anti-symmetric
- Transitive.

Notation: $x, y \in R$ xRy is denoted $x \preceq y$.

Standard example: $(\mathbb{Z}, \leq)$

set ↗ ↖ relation.

x is "related to" y if $x \leq y$.

- Reflexive: $x \leq x$
- Anti-symmetric: $x \leq y$ and $y \leq x \Rightarrow x = y$.
- Transitive: $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

Example: $(\mathbb{Z}^+, \preceq_P)$ $x \preceq_P y$ iff $\exists\, n \in \mathbb{N}$ $x^n = y$.

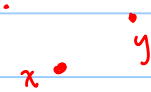
- Reflexive:
- Anti-symmetric:
- Transitive:

Hasse Diagrams :

Clean way to depict partial order $(A, \preceq)$.

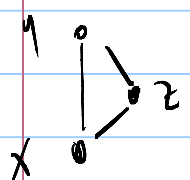
* If $x \preceq y$ the x appears below y

↳ converse not necessarily true.



x & y could be incomparable.

* A line from x to y if $x \preceq y$ and $\nexists z$ $x \preceq z$ and $z \preceq y$.



$(\underbrace{\{2,4,8,16,32,64\}}, \preceq_p)$

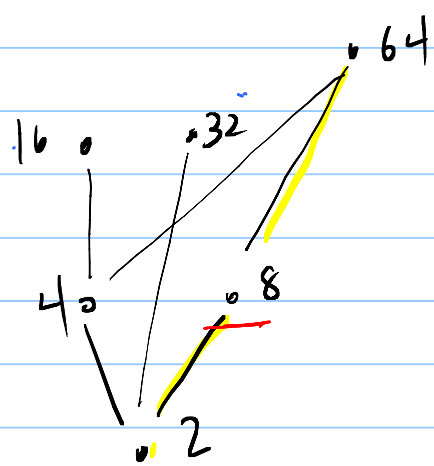
↳ $x \preceq_p y$: y a power of x

Is there a line between 2 & 16?

$2 \preceq 16$ $2^4 = 16$.

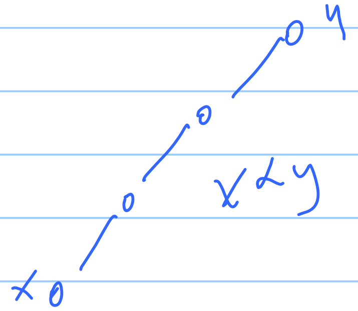
4 & 32?

$\cancel{4 \overset{?}{\preceq} 32}$



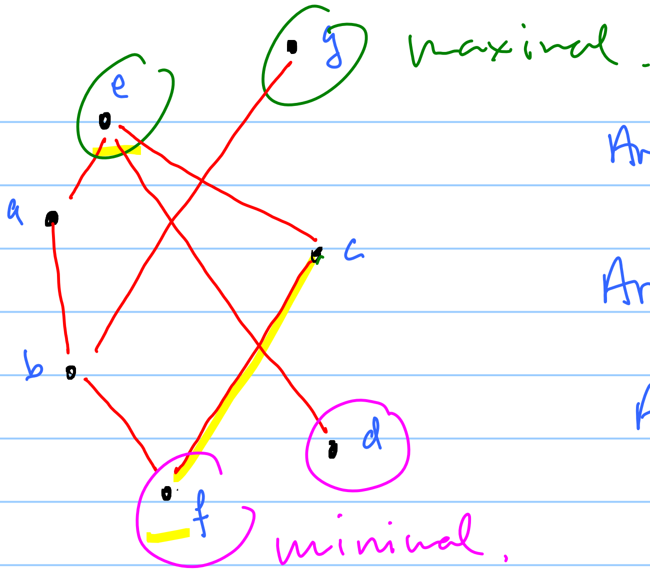
Maximal element : x

$\nexists y$ $x \preceq y$



Minimal element : x

$\nexists y$ $y \preceq x$.



Are e & f comparable

$f \preceq e$.

Are e and f comparable?

$f \preceq e$.

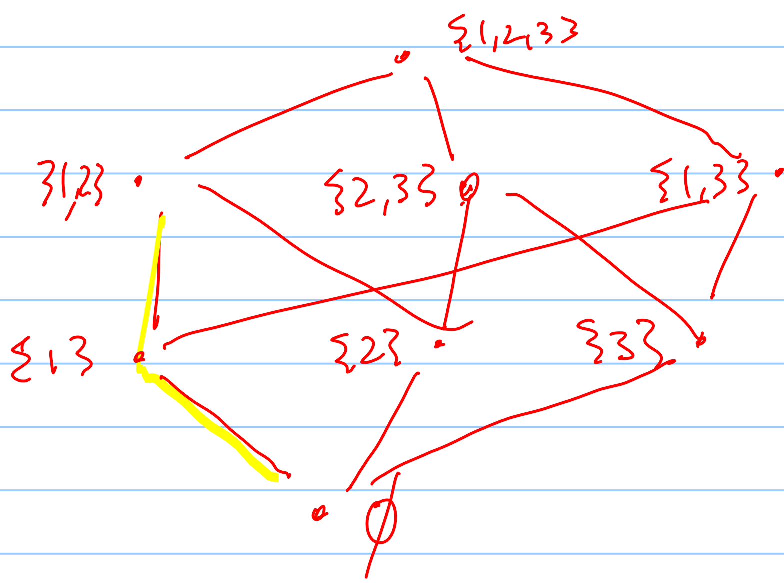
Are d + g comparable?

no.

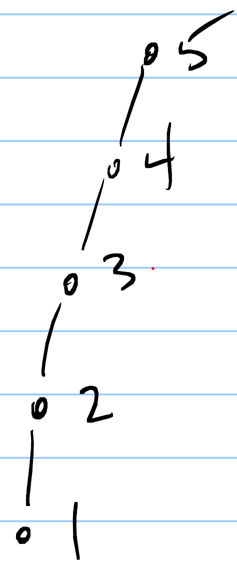
Minimal Elements: x is minimal if there is no y $y \preceq x$

Maximal Elements: x maximal if there is no y $x \preceq y$.

$A = \{1, 2, 3\}$ $\qquad (P(A), \subseteq)$.



Hasse Diagram for $(\{1,2,3,4,5\}, \leq)$



Strict Orders

Note Title 2/27/2015

Partial Order $\preceq$	Strict Order $\prec$
* Reflexive	* Anti-reflexive
* Anti-symmetric	* ~~Anti-symmetric~~
* Transitive	* Transitive.

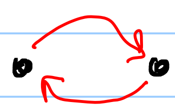
Note: If a relation R is anti-reflexive and transitive then it is also anti-symmetric.

(Anti-Reflexive) $\wedge$ (Transitive) $\longrightarrow$ (Anti-Symmetric)

contra-positive:

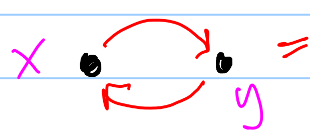
$\neg$ (Anti-Symmetric) $\wedge$ (Transitive) $\rightarrow \neg$ (Anti-Reflexive)

Anti-Symmetric means that you never have this pattern:



$\Leftarrow$

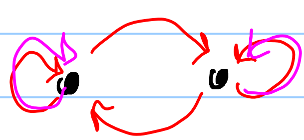
$\neg$ (Anti-Symmetric) means that you have this pattern somewhere

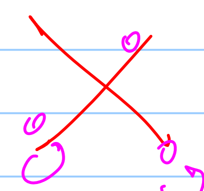


=

If it is transitive then you also have:

$xRy \wedge yRx \Rightarrow xRx.$





$\Rightarrow$ $\neg$ (Anti-reflexive).

Examples of ~~partial~~ strict orders:

$(\mathbb{Z}, <)$

* Anti-reflexive $x \not< x$

* Anti-symmetric.

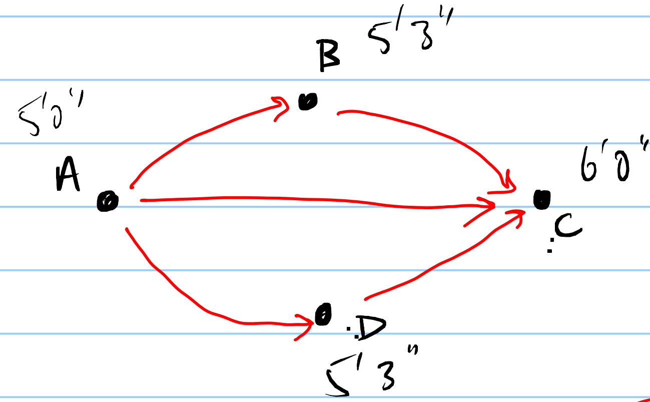
$\underbrace{x<y \wedge y<x}_{\text{false for every pair.}} \Rightarrow x=y$

* Transitive: $x<y \wedge y<z \to x<z$.

Group of people. $x < y$ if x is shorter than y.

Height

A : 5'0"
B : 5'3"
C : 6'0"
D : 5'3"



$P \to q$
$\parallel$
F

$x R y \wedge y R x \Rightarrow x = y$.

* Anti-reflexive "x is shorter than x"? never.

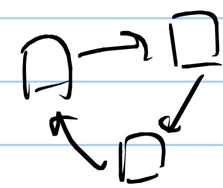
* (Anti-Symmetric) x is shorter than y, y is shorter than x $\} \to x=y$.

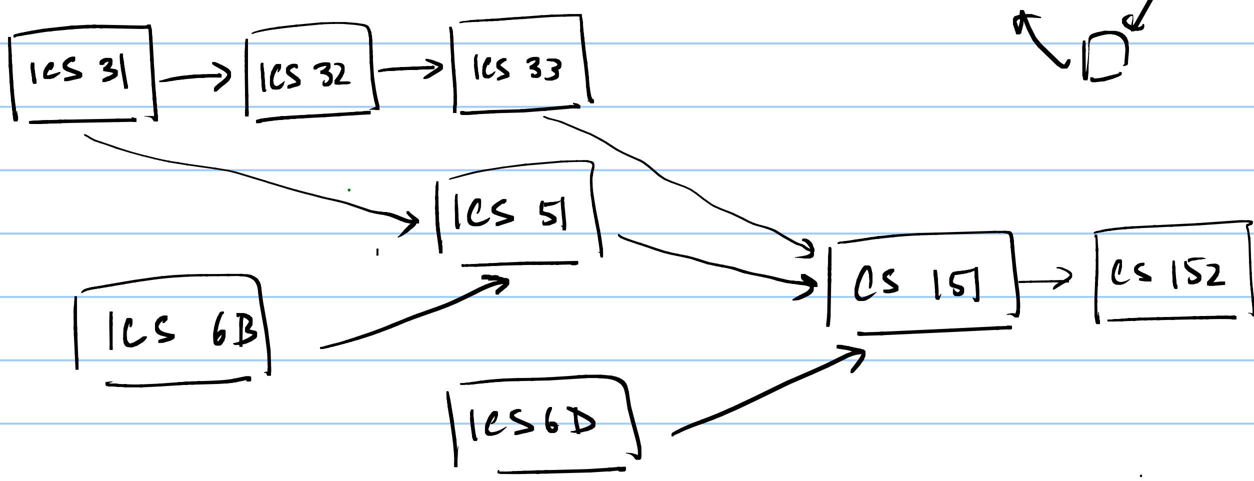
* Transitive x is shorter than y, y is shorter than z $\} \Rightarrow$ x is shorter than z.

Total order? Not if there are two people in the group who have the same height.

Strict order: useful for representing precedence relationships.

Vertices: set of tasks.
Relation: x must be completed before y begins.





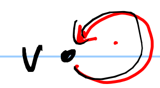
ICS 32 has to be completed before CS 151, but this fact is not explicitly mentioned in the catalog.

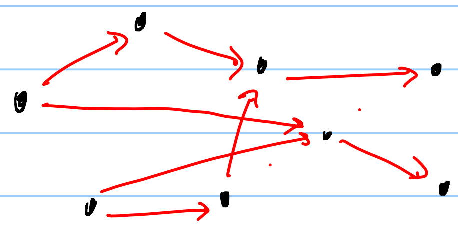
Prequisites are represented by an acyclic graph that is not necessarily transitive.

A directed graph is acyclic if it does not have any positive length cycles.

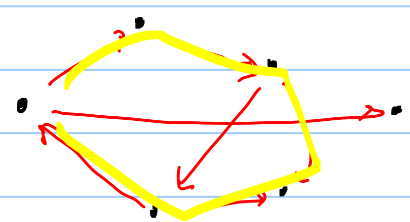
Directed Acyclic Graph ≡ "DAG".

Note: $\langle v \rangle$ is a cycle of length 0 (no edges)
$\langle v, v \rangle$ is a cycle of length 1 (one edge).
only if there is a "self loop" at vertex v.





acyclic

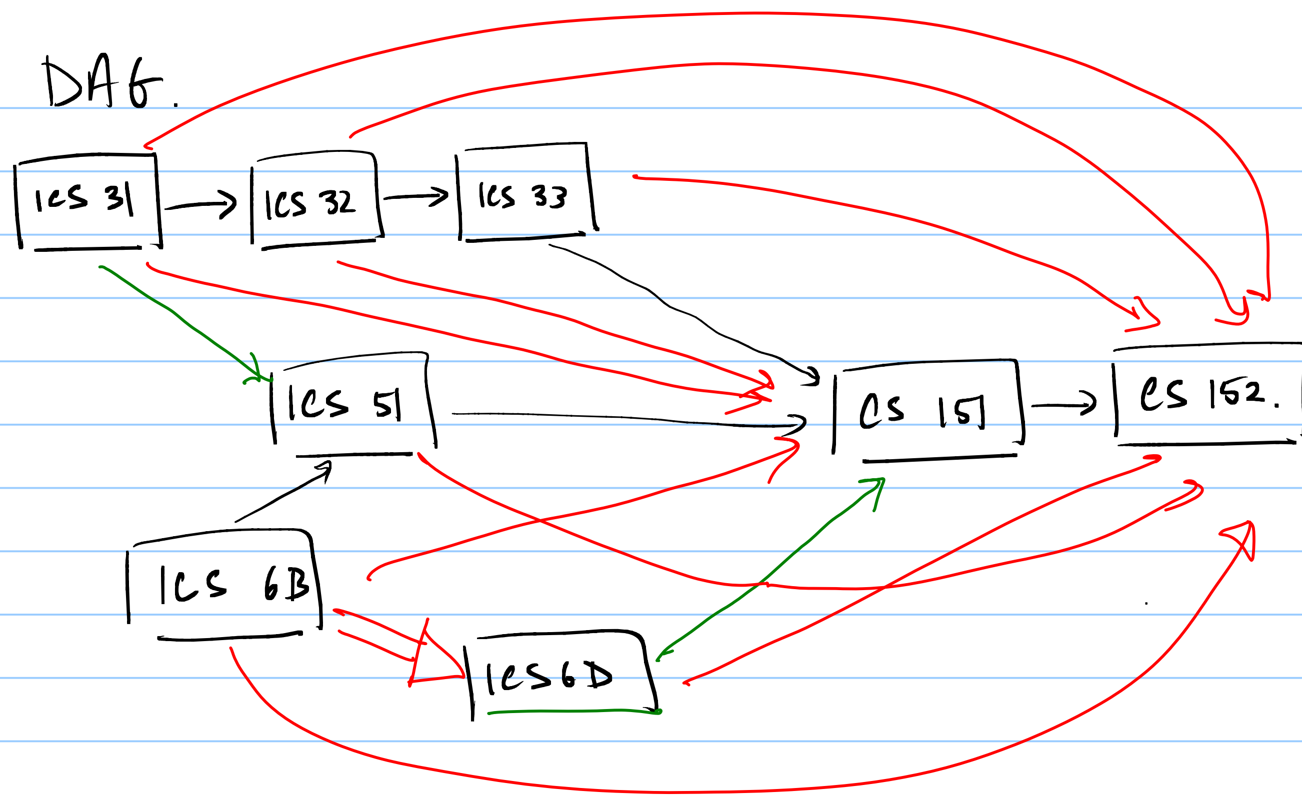


Graph representing prerequisite structure is a DAG G.

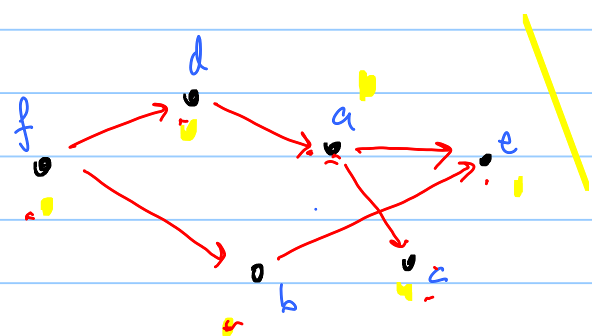
Course x before Course y $\Longleftrightarrow$ (x,y) is in G^+ transitive closure of G^+.

Theorem: G is a directed acyclic graph if and only if G^+ is a strict order

DAG.



A topological sort of a DAG is an ordering of the vertices such that for every edge (u,v), u comes before v in the ordering.



f, d, a, c, b, e Yes.

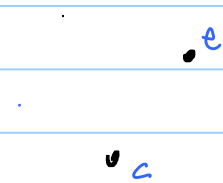
f, c, d, b, a, e

f, b, d, a, e, c Yes!

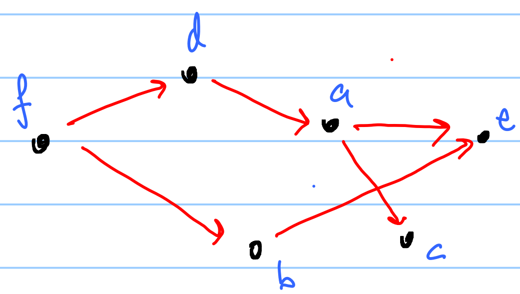
A topological sort for a DAG is not necessarily unique.

To find a topological sort, keep removing vertices that are minimal (i.e. in-degree = 0).

Order of removal is a topological sort.



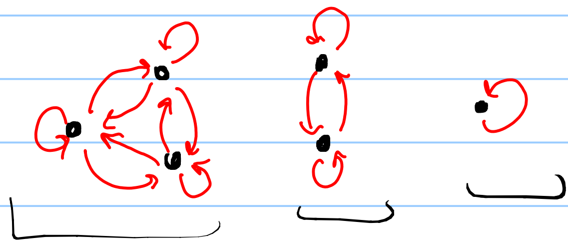
f, b, d, a, c, e.



Equivalence Relations.

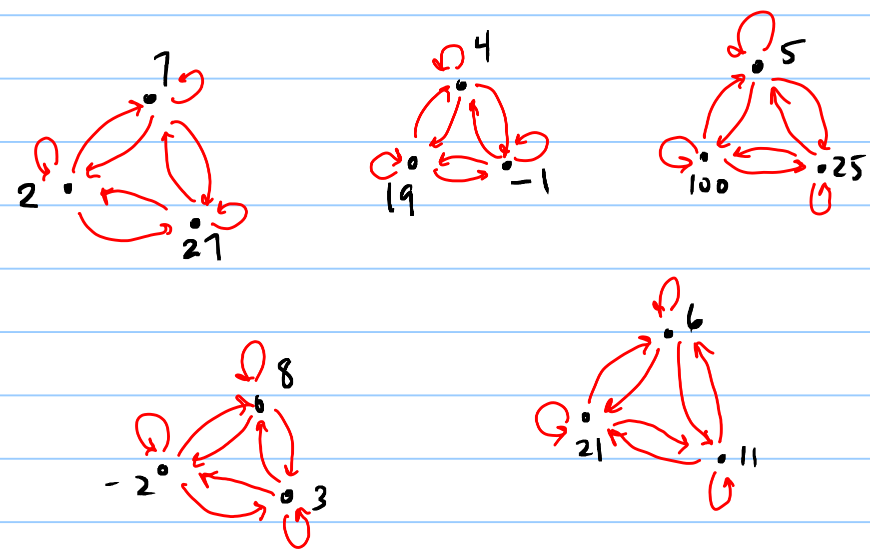
A Relation R on a set A is an equivalence relation if it is:
- reflexive
- symmetric $\Leftrightarrow$
- transitive.

if xRy then $x \sim y$. $x \sim y$.



Example: domain $\mathbb{Z}$

$x \sim y$ iff $x - z$ is a multiple of 5.



$x \sim y$ iff $(x-y) = 5n$ for integer n.

Reflexive: $\underline{x - x} = 5 \cdot 0$

Symmetric: $\overset{x \sim y}{(x-y) = 5n} \Rightarrow \overset{y \sim x}{\overline{(y-x) = -5n = 5(-n)}}$.

Transitive: $\begin{matrix} \| (x-y) = 5n \\ \| (y-z) = 5m \end{matrix} \Rightarrow \underline{(x-z)} = (x-y) + (y-z)$
$= 5n + 5m$
$= 5(n+m)$.

Group of people $x \sim y$ if x and y have the same birth day.

Reflexive: $x \sim x$. ✓

Symmetric $x \sim y \to y \sim x$ ✓

Transitive: $x \sim y \wedge y \sim z \Rightarrow x \sim z$ ✓

Equivalence Class of x: $[x] = \{y \mid x \sim y\}$.

The set of distinct equivalence classes forms a partition of the underlying set.

How many equivalence classes for the birthday equivalence relation?

... as many as $\underline{366}$.

What are the equivalence classes of

$x \sim y$ iff $(x-y)$ is a multiple of 5.

$[0]$ $[1]$ $[2]$ $[3]$ $[4]$