# Notes on Matrix Multiplication and the Transitive Closure

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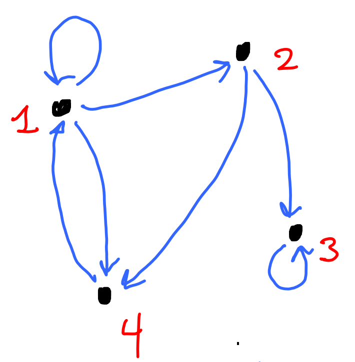
An $n \times m$ **matrix** over a set S is an array of elements from S with n rows and m columns. Each element in a matrix is called an **entry**. The entry in row $i$ and column $j$ is denoted by $A_{i,j}$. A matrix is called a **square matrix** if the number of rows is equal to the number of columns. Here are some examples of matrices.

$$\begin{bmatrix} 2 & -7 & 4 \\ 3 & 16 & -1 \end{bmatrix} \qquad \begin{bmatrix} 3.1 & -1.6 \\ -.7 & 4.2 \\ 42 & 5.1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

2 x 3 matrix over $\mathbb{Z}$ — 3x2 matrix over $\mathbb{R}$ — 2x2 matrix over $\{0,1\}$

A directed graph with n vertices can be represented by an $n \times n$ matrix called the **adjacency matrix** for the graph. If matrix A is the adjacency matrix for a graph G then $A_{i,j} = 1$ if there is an edge from vertex i to vertex j in G. Otherwise, $A_{i,j} = 0$. Here is an example of a directed graph and its adjacency matrix.




Directed Graph G

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix for G.

If mathematical operations, such as addition and multiplication, are defined on a set S, then matrix addition and multiplication can be defined for matrices over the set S. A **Boolean matrix** is a matrix whose entries are from the set $\{0, 1\}$. Boolean addition and multiplication are used in adding and multiplying entries of a Boolean matrix. We define matrix addition and multiplication for square Boolean matrices because those operations can be used to compute the transitive closure of a graph. Matrix multiplication and addition can be defined for general rectangular matrices over other sets such as the real numbers and are useful operations in other contexts such as scientific applications or computer graphics.

The product of two square matrices, A and B, is well defined only if A and B have the same number of rows and columns. Computing the dot product of a row in A and a column in B is an important step in computing the product of A and B.

If A and B are $n \times n$ matrices, the **dot product** of row i of A and column j of B is the sum of the product of each entry in row i from A with the corresponding entry in column j from B:

$$A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,n}B_{n,j}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Dot product of row 2 of A and column 3 of B:

$$\begin{array}{r} 0 \\ \times\ 1 \\ \hline 0 \end{array} + \begin{array}{r} 1 \\ \times\ 0 \\ \hline 0 \end{array} + \begin{array}{r} 1 \\ \times\ 1 \\ \hline 1 \end{array} = 1.$$

If A and B are $n \times n$ matrices over the integers, then the **matrix product** of A and B, denoted $AB$ or $A \cdot B$, is another $n \times n$ matrix such that $(AB)_{i,j}$ is the result of taking the dot product of row i of matrix A and column j of matrix B.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & \boxed{0} & 1 \end{bmatrix}}$$

Entry in row 3, column 2 of C
= dot product of row 3 from A
and column 2 from B.

Matrix multiplication is associative, meaning that if A, B, and C are all $n \times n$ matrices, then $A(BC) = (AB)C$. However, matrix multiplication is not commutative because in general $AB \neq BA$. The $k^{th}$ power of a matrix A is the product of k copies of A:

$$A^k = \underbrace{A \cdot A \cdots A}_{\text{k times}}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

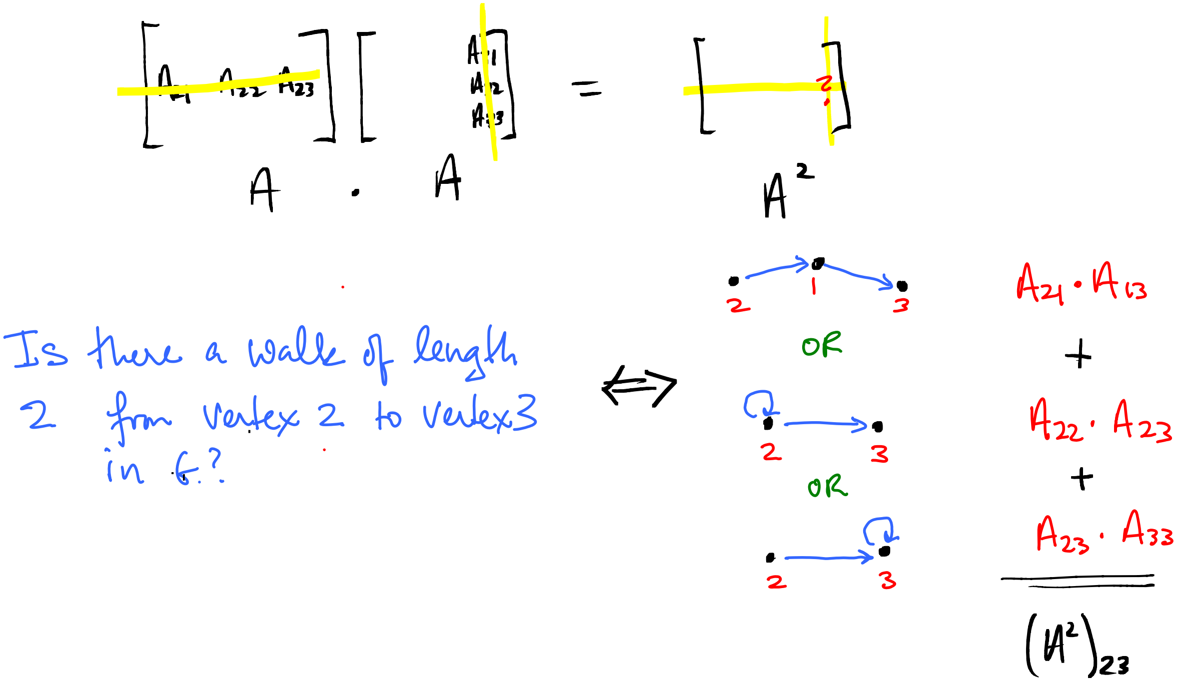
$$A^3 = A \cdot A \cdot A = A^2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^4 = A \cdot A \cdot A \cdot A = A^3 \cdot A \ \ldots \text{etc.}$$

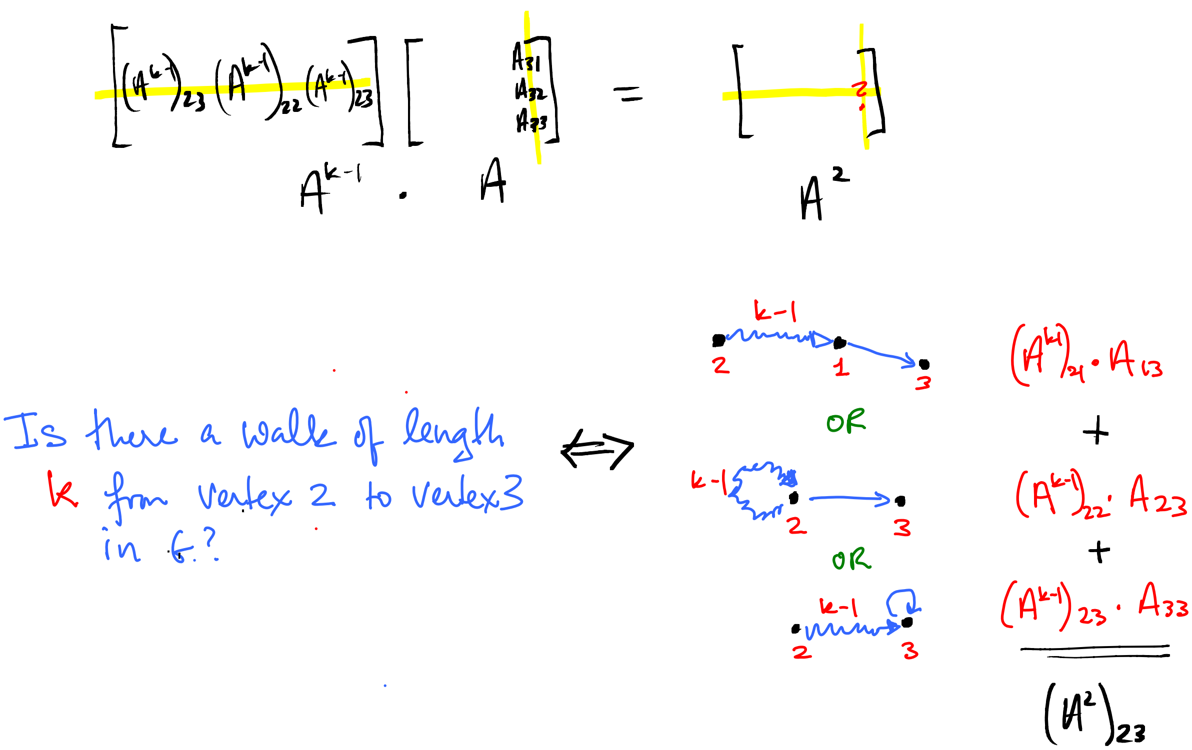
The relationship between the powers of a graph and the powers of its adjacency matrix is defined in the following theorem.

**Theorem 1.** *Let G be a directed graph with n vertices and let A be the adjacency matrix for G. Then for any* $k \geq 1$*,* $A^k$ *is the adjacency matrix of* $G^k$*, where Boolean addition and multiplication are used to compute* $A^k$.

The proof of the theorem is given below. Here is some intuition why it works. Condider just multiplying $A \cdot A$. The condition for whether there is a walk of length two from vertex 2 to vertex 3 is whether there is some other vertex, say $x$, such that $(2, x)$ is an edge and $(x, 3)$ is an edge. This is same as the condition that there is an $x$ such that $A_{2,x}A_{x,3} = 1$. If such an $x$ exists, then the entry in $A^2$ in row 2 and column 3 is 1. This idea is illustrated below:




Now, what are the conditions for there being an edge from vertex 2 to vertex 3 in $G^k$? This means there is a walk of length $k$ from vertex 2 to vertex 3 in $G$. Look at the second to last vertex in this path. Call it $x$. There must be a path of length $k-1$ from 2 to $x$ and an edge from $x$ to 3. This is the same as the condition that in matrix $A^{k-1}$, entry $(2,3)$ is 1 (i.e., $(A^{k-1})_{2,x}$) and $A_{x,3} = 1$. If such an $x$ exists, then the entry in $A^k$ in row 2 and column 3 is 1. This idea is illustrated below:



**Proof:** The proof is by induction on $k$. For the base case, $k = 1$. By definition $G^1 = G$, and $A^1 = A$ is the adjacency matrix for $G$.

Now assume that $A^{k-1}$ is the adjacency matrix for $G^{k-1}$, and prove that $A^k$ is the adjacency matrix for $G^k$. Since $A^{k-1}$ is the adjacency matrix for $G^{k-1}$, $(A^{k-1})_{i,j}$ is 1 if and only if there is a walk in graph $G$ of length $k-1$ from vertex $i$ to vertex $j$. We will show that $(A^k)_{i,j} = 1$ if and only if there is a walk of length $k$ in $G$ from vertex $i$ to vertex $j$.

Suppose that $(A^k)_{i,j} = 1$. Since $A^k = A^{k-1} \cdot A$, by definition of matrix multiplication,

$$(A^k)_{i,j} = (A^{k-1})_{i,1}A_{1,j} + (A^{k-1})_{i,2}A_{2,j} + \cdots + (A^{k-1})_{i,n}A_{n,j}$$

Since $(A^k)_{i,j} = 1$, there is at least one $x$ in the range from 1 through $n$ such that $(A^{k-1})_{i,x} \cdot A_{x,j} = 1$. For that value of $x$, $(A^{k-1})_{i,x} = 1$ and $A_{x,j} = 1$. Thus, there is a walk of length $k-1$ in $G$ from vertex $i$ to vertex $x$ and an edge in $G$ from vertex $x$ to vertex $j$ which means that there is a walk of length $k$ in $G$ from vertex $i$ to vertex $j$.

Now suppose that there is a walk of length $k$ in $G$ from vertex $i$ to vertex $j$. Let $x$ be the last vertex visited on the walk before $j$ is reached. There is a walk of length $k-1$ from $i$ to $x$ in $G$ and an edge from vertex $x$ to vertex $j$ in $G$. By induction, $(A^{k-1})_{i,x} = 1$ and $A_{x,j} = 1$. Therefore, by definition of Boolean matrix multiplication $(A^k)_{i,j} = 1$. ■

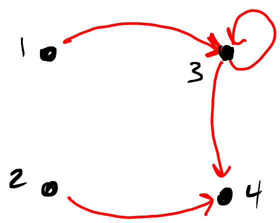
The sum of two matrices is well defined if they have the same number of rows and the same number of columns.

If $A$ and $B$ are two $m \times n$ matrices, then the **matrix sum** of $A$ and $B$, denoted $A + B$, is also an $m \times n$ matrix such that $(A + B)_{i,j} = A_{i,j} + B_{i,j}$.

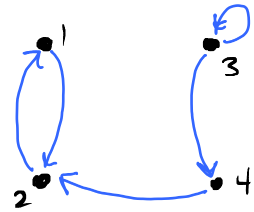
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} = \begin{bmatrix} (A_{11}+B_{11}) & (A_{12}+A_{12}) & (A_{13}+B_{13}) \\ (A_{21}+B_{21}) & (A_{22}+B_{22}) & (A_{23}+B_{23}) \end{bmatrix}$$

The union of two graphs defined on the same set of vertices is a single graph whose edges are the union of the edge sets of the two graphs. Graph union can be computed using matrix addition:

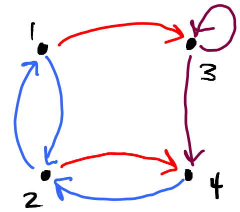
**Theorem 2.** *Let $G$ and $H$ be two directed graphs with the same vertex set. Let $A$ be the adjacency matrix for $G$ and $B$ the adjacency matrix for $H$. Then the adjacency matrix for $G \cup H$ is $A + B$, where Boolean addition used on the entries of matrices $A$ and $B$.*




$G$




$H$




$G \cup H$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Finally, Boolean matrix multiplication and addition can be put together to compute the adjacency matrix A¡sup¿+¡/sup¿ for $G^+$, the transitive closure of $G$:

$$\begin{aligned} G^+ &= G^1 \cup G^2 \cup \cdots \cup G^n \\ A^+ &= A^1 + A^2 + \cdots + A^n \end{aligned}$$