Principles of Quantum Computers for 1-qubit.

Note Title 1/11/2015

Qubit is a 2-state quantum system (for example the spin of an electron that can be up or down).

Represent the two states as $|0\rangle$ and $|1\rangle$

Superposition principle:
the system can be partially in different states at the same time:

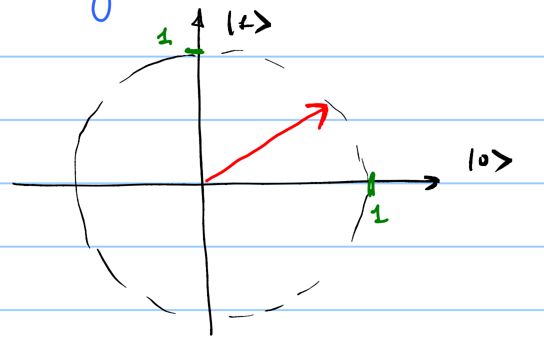
$$\alpha_0|0\rangle + \alpha_1|1\rangle$$

α_0 and α_1 are called amplitudes. They represent the extent to which the system is in state $|0\rangle$ or $|1\rangle$.

α_0 and α_1 can in general be complex numbers, but to keep the discussion simpler, we will assume that they are real (possibly negative) numbers. We require that

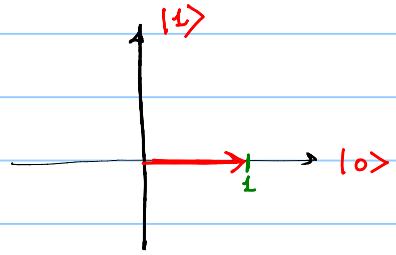
$$(\alpha_0)^2 + (\alpha_1)^2 = 1.$$

We can also represent the state $\alpha_0|0\rangle + \alpha_1|1\rangle$ by a vector (α_0, α_1).



We can visualize the vector in the plane.
$\alpha_0^2 + \alpha_1^2 = 1$ means that the vector is on the unit circle.

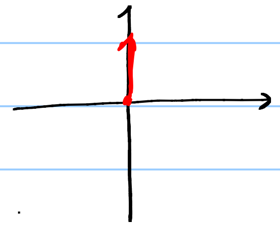
Here are some examples of states:



$|0\rangle$

$\alpha_0 = 1, \; \alpha_1 = 0$

$(1, 0)$



$|1\rangle$

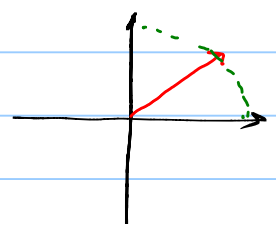
$\alpha_0 = 0 \quad \alpha_1 = 1$

$(0, 1)$

The vectors for $|0\rangle$ + $|1\rangle$ are perpendicular.
Mathematically, two vectors (α_0, α_1) and (β_0, β_1) are perpendicular if $\alpha_0 \cdot \beta_0 + \alpha_1 \cdot \beta_1 = 0$.

Verify mathematically that $(1, 0)$ + $(0, 1)$ are perpendicular.

Another state is:



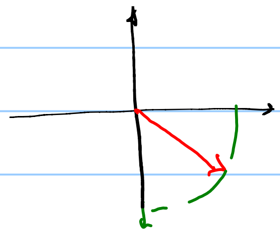
if $\alpha_0 = \alpha_1 \quad \alpha_0 > 0, \; \alpha_1 > 1.$
and $\alpha_0^2 + \alpha_1^2 = 1$
what is α_0?

$$\alpha_0^2 + \alpha_1^2 = \alpha_0^2 + \alpha_0^2 = 2\alpha_0^2 = 1$$
$$\alpha_0^2 = 1/2$$
$$\alpha_0 = 1/\sqrt{2}.$$

$1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle$

$(1/\sqrt{2}, 1/\sqrt{2})$ ↳ called $|+\rangle$.

One more example of a qubit state:



$\alpha_0 = -\alpha_1$ $\qquad$ $\alpha_0 > 0$

$\alpha_1 < 0.$

$1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle = |-\rangle$

$(1/\sqrt{2}, -1/\sqrt{2}).$

Verify that $|+\rangle$ is perpendicular to $|-\rangle$

$(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$

Suppose we have $\frac{1}{\sqrt{3}}|0\rangle + \alpha_1 |1\rangle$.

What are the choices for α_1?

$$(1/\sqrt{3})^2 + (\alpha_1)^2 = 1.$$

$$(1/3) + \alpha_1^2 = 1$$

$$(\alpha_2)^2 = 2/3$$

$$\alpha_2 = \pm\sqrt{2/3}$$

(If we allow α_1 to be complex then there are more choices, but we are sticking to real numbers for now).

Any state $\alpha_0|0\rangle + \alpha_1|1\rangle$

can be written as $\beta_0|+\rangle + \beta_1|-\rangle$

if $(\alpha_0)^2 + (\alpha_1)^2 = 1$, then $(\beta_0)^2 + (\beta_1)^2 = 1.$

$$\beta_0\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) + \beta_1\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) = \alpha_0|0\rangle + \alpha_1|1\rangle$$

$\frac{1}{\sqrt{2}}(\beta_0 + \beta_1) = \alpha_0$ $\qquad$ $\beta_0 = (\alpha_0 + \alpha_1)/\sqrt{2}.$

$\frac{1}{\sqrt{2}}(\beta_0 - \beta_1) = \alpha_1$ $\qquad$ $\beta_1 = (\alpha_0 - \alpha_1)/\sqrt{2}.$

Express the state $\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$ as $\beta_0|+\rangle + \beta_1|-\rangle$

Measurement Principle:

If we measure the qubit $\alpha_0|0\rangle + \alpha_1|1\rangle$
to see if it is $|0\rangle$ or $|1\rangle$.
we get A) $|0\rangle$ with prob $(\alpha_0)^2$
B) $|1\rangle$ with prob $(\alpha_1)^2$.

why $(\alpha_0)^2 + (\alpha_1)^2 = 1$. probabilities must sum to 1!

If A) happens, the state is $|0\rangle$ $(1,0)$ afterwards.
If B) happens, the state is $|1\rangle$ $(0,1)$ afterwards.

There is no memory of what the state was prior to measurement.
Measurement changes the state.

If we measure $|+\rangle$ what is the prob of $|0\rangle$? or $|1\rangle$?

An alternative measurement:

$$\alpha_0|0\rangle + \alpha_1|1\rangle = \beta_0|+\rangle + \beta_1|-\rangle$$

Could also measure if the state is $|+\rangle$ or $|-\rangle$
A) Outcome is $|+\rangle$ w/ prob $(\beta_0)^2$
B) Outcome is $|-\rangle$ w/ prob $(\beta_1)^2$

"measure in the $|+\rangle$ $|-\rangle$ basis".

Afterwards, state is $|+\rangle$ or $|-\rangle$ depending on the outcome of the measurement.

If we measure the state $|1\rangle$ in the $|+\rangle$ or $|-\rangle$ basis, then what is the probability of each outcome?

Final principle: "Unitary evolution".

Any change to a quantum system over time must satisfy:

If two vectors/states are perpendicular before, they will also be perpendicular afterwards.

Change to a quantum system (transformation) can be described by matrix multiplication:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 a + \alpha_1 \cdot c \\ \alpha_0 b + \alpha_1 b \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

How do these change $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$?

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$