Note Title 1/28/2015

Two qubit systems:

4 possible states $|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

Any quantum state is a "superposition" of these four possibilities:

Can be summarized! $(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})$.

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

The α's are called "amplitudes". As with 1-qubit systems they can be complex numbers, but to keep things simpler, we will assume they are real (possibly negative) numbers.

If we measure both qubits to see if they are 0 or 1, there are four possible outcomes. The probability of getting $|00\rangle$ is $(\alpha_{00})^2$. (Same for all the other possible outcomes).

Since the probabilities have to sum to 1, we have:

$$(\alpha_{00})^2 + (\alpha_{01})^2 + (\alpha_{10})^2 + (\alpha_{11})^2 = 1.$$

If the outcome of a measurement is $|00\rangle$ then the state "collapses" to $|00\rangle$.

Result of measuring a 2-qubit state.

$(\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})$

with prob $(\alpha_{00})^2$ w/ prob $(\alpha_{01})^2$ prob: $(\alpha_{10})^2$ prob $(\alpha_{11})^2$

$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$

$|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

For example : $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$.

If we measure, we get each of the four outcomes with probability $(\frac{1}{2})^2 = \frac{1}{4}$.

Note that : $(\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = 1$.

What if we measure only the first qubit?
The second qubit remains uncertain.

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

prob $P_0 = (\alpha_{00})^2 + (\alpha_{01})^2$ prob $P_1 = (\alpha_{10})^2 + (\alpha_{11})^2$

$$\underbrace{\frac{\alpha_{00}}{\sqrt{P_0}}|00\rangle + \frac{\alpha_{01}}{\sqrt{P_0}}|01\rangle} \qquad \frac{\alpha_{10}}{\sqrt{P_1}}|10\rangle + \frac{\alpha_{11}}{\sqrt{P_1}}|11\rangle$$

[to ensure ampl squares sum up to 1.]

1st qubit is definitely 1, second qubit is uncertain.

Check: $\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2=1.$

Example:
Measure first qubit

$$\frac{1}{3}|00\rangle+\frac{2}{3}|01\rangle+\frac{1}{3}|10\rangle+\frac{1}{\sqrt{3}}|10\rangle$$

$\text{Prob}=\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{5}{9}$ $\qquad$ $\text{Prob}=\left(\frac{1}{3}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2=\frac{4}{9}.$

$$\frac{1/3}{\sqrt{5/9}}|00\rangle+\frac{2/3}{\sqrt{5/9}}|01\rangle \qquad\qquad \frac{\frac{1}{3}}{\sqrt{4/9}}|10\rangle+\frac{\frac{1}{\sqrt{3}}}{\sqrt{4/9}}|11\rangle$$

$$=\frac{1}{\sqrt{5}}|00\rangle+\frac{2}{\sqrt{5}}|01\rangle \qquad\qquad =\frac{1}{2}|10\rangle+\frac{\sqrt{3}}{2}|11\rangle$$

Check $(1/\sqrt{5})^2+(3/\sqrt{5})^2=1.$ $\qquad$ Check: $\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=1.$

Quantum Entanglement:

Example:
Measure the first qubit.

$$\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|11\rangle \qquad (1/\sqrt{2},0,0,1/\sqrt{2}).$$

$P_0=1/2$ $\qquad$ $P_1=1/2.$

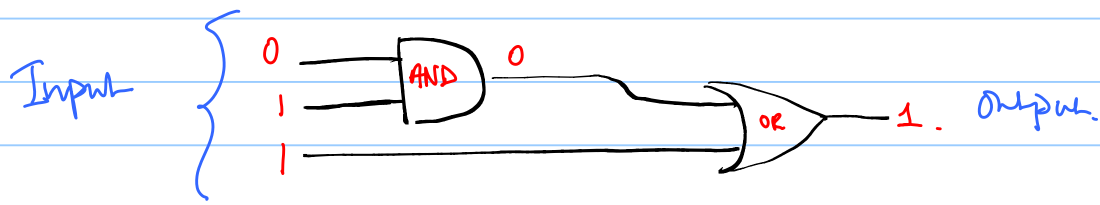
$$\cancel{\frac{1/\sqrt{2}}{\sqrt{1/2}}}|00\rangle \qquad\qquad \frac{1/\sqrt{2}}{(1/2)^2}|11\rangle$$

After measuring the first qubit, the state of the second qubit is also determined and is always the same as the first qubit.

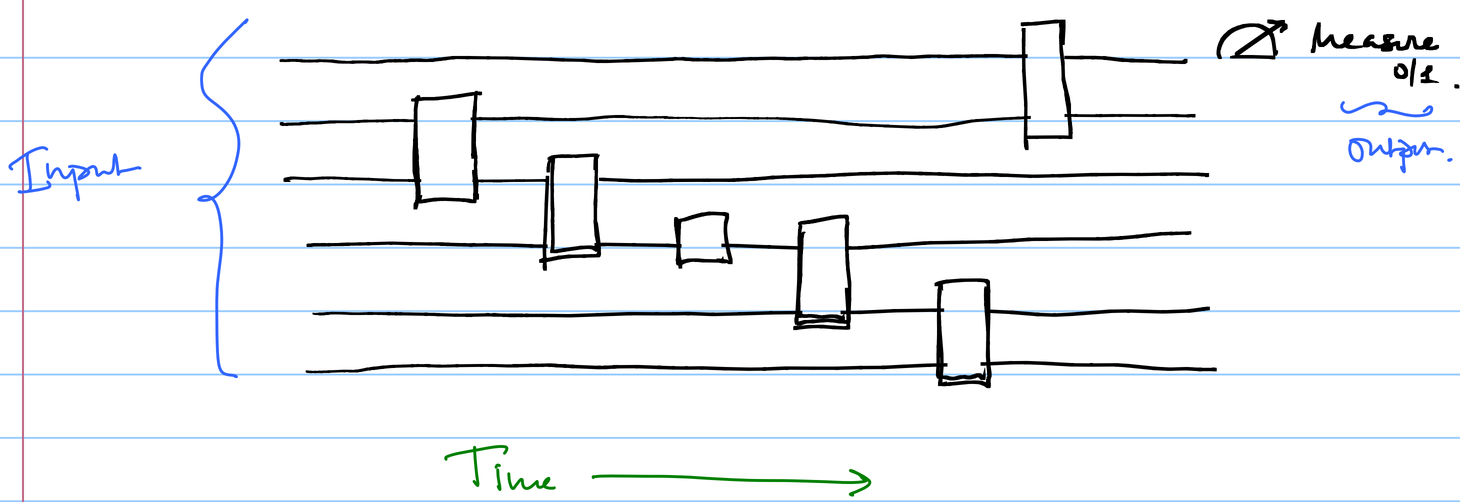
This has been done experimentally in which the two quibits are separated by kilometers. When the first qubit is measured the second qubit changes state instantly. The second qubit is measured before a message traveling at the speed of light could go from the first qubit to the second.

In the original state, the two qubits are "entangled". This entanglement is a uniquely quantum phenomenon. Quantum entanglement is an important ingredient in quantum computing.

__Circuits__: Classical circuits look like:



Quantum circuits look like:



We have already seen 1-qubit gates:

$$H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A 2-qubit gate is a 4x4 matrix.

Here is an example called CNOT:

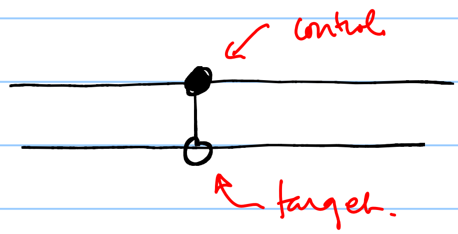
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |01\rangle$$
$$|10\rangle \longrightarrow |11\rangle$$
$$|11\rangle \longrightarrow |10\rangle$$

The first qubit is the "control" bit. 2nd qubit is the target.
If the control bit is 0, then no change.
If the control bit is 1, then the target bit is flipped.

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle.$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|11\rangle + \alpha_{11}|10\rangle$$

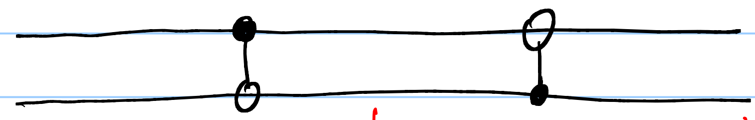
$$\alpha_{00}|00\rangle + \alpha_{01}|01| + \alpha_{11}|10\rangle + \alpha_{10}|11\rangle$$



Sample Circuit:

$\alpha|0\rangle + \beta|1\rangle$

$|0\rangle$



$|0\rangle$

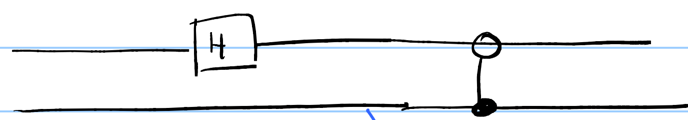
$\alpha|0\rangle + \beta|1\rangle$

$\alpha|00\rangle + \beta|10\rangle$

$\alpha|00\rangle + \beta|11\rangle$ $\qquad$ $\alpha|00\rangle + \beta|01\rangle$

(qubit transfer)

Another Example



$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

target $\qquad$ control.

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle\right) = \frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|11\rangle$$

$\Downarrow$ CNOT

$$\underbrace{\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle}_{\substack{\text{control bit} = 0 \\ \text{no change}}} + \underbrace{\frac{1}{2}|11\rangle - \frac{1}{2}|01\rangle}_{\substack{\text{control bit} = 1 \\ \text{flip target}}}$$

H: $|0\rangle \longrightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$\quad |1\rangle \longrightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$