Set 8: Inference in First-order logic
Outline

◊ Reducing first-order inference to propositional inference
◊ Unification
◊ Generalized Modus Ponens
◊ Forward and backward chaining
◊ Logic programming
◊ Resolution
Universal instantiation (UI)

- Every instantiation of a universally quantified sentence is entailed by it:
  \[ \forall v \alpha \\Rightarrow \text{Subst}(\{v/g\}, \alpha) \]
  for any variable \( v \) and ground term \( g \)

- E.g., \( \forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x) \) yields:

  \begin{align*}
  &\text{King}(John) \land \text{Greedy}(John) \Rightarrow \text{Evil}(John) \\
  &\text{King}(Richard) \land \text{Greedy}(Richard) \Rightarrow \text{Evil}(Richard) \\
  &\text{King}(\text{Father}(John)) \land \text{Greedy}(\text{Father}(John)) \Rightarrow \text{Evil}(\text{Father}(John))
  \end{align*}

  Obtained by substituting \( \{x/John\}, \{x/Richard\} \) and \( \{x/\text{Father}(John)\} \)
Existential instantiation (EI)

- For any sentence $\alpha$, variable $v$, and constant symbol $k$ that does not appear elsewhere in the knowledge base:

$$\exists v \alpha$$

$$\text{Subst}\{\{v/k\}, \alpha\}$$

- E.g., $\exists x \ Crown(x) \land OnHead(x, John)$ yields:

$$Crown(C_1) \land OnHead(C_1, John)$$

provided $C_1$ is a new (not used so far) constant term, called a Skolem constant

- Skolemization: $\exists$ elimination
  - $\forall x \exists y \ Loves(y, x)$
  - Incorrect inference: $\forall x \ Loves(A, x)$ – $y$ may be different for each $x$
  - Correct inference: $\forall x \ Loves(f(x), x)$
Suppose the KB contains just the following:

\[
\forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x)
\]
King(John)
Greedy(John)
Brother(Richard,John)

- Instantiating the universal sentence in all possible ways, we have:
  King(John) \land Greedy(John) \Rightarrow Evil(John)
  King(Richard) \land Greedy(Richard) \Rightarrow Evil(Richard)
  King(John)
  Greedy(John)
  Brother(Richard,John)

- The new KB is **propositionalized**: proposition symbols are

  King(John), Greedy(John), Evil(John), King(Richard), etc.
Reduction contd.

• Every FOL KB can be propositionalized so as to preserve entailment
  – A ground sentence is entailed by new KB iff entailed by original KB

• Idea: propositionalize KB and query, apply resolution, return result

• Problem: with function symbols, there are infinitely many ground terms,
  – e.g., Father(Father(Father(John))))
Reduction contd.

Theorem: Herbrand (1930). If a sentence \( \alpha \) is entailed by an FOL KB, it is entailed by a **finite** subset of the propositionalized KB

**Idea:** For \( n = 0 \) to \( \infty \) do
- create a propositional KB by instantiating with depth=\( n \) terms
- see if \( \alpha \) is entailed by this KB

**Problem:** works (will terminate) if \( \alpha \) is entailed, loops forever if \( \alpha \) is not entailed

Theorem: Turing (1936), Church (1936) Entailment for FOL is **semidecidable** (algorithms exist that say yes to every entailed sentence, but no algorithm exists that also says no to every non-entailed sentence.)
Problems with propositionalization

- Propositionalization seems to generate lots of irrelevant sentences.

- E.g., from:

  \[ \forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x) \]
  \[ \text{ King}(\text{John}) \]
  \[ \forall y \text{ Greedy}(y) \]
  \[ \text{ Brother}(\text{Richard}, \text{John}) \]

- Given query “\text{ Evil}(x)” it seems obvious that \( \text{ Evil}(\text{John}) \), but propositionalization produces lots of facts such as \( \text{ Greedy}(\text{Richard}) \) that are irrelevant.

- With \( p \) \( k \)-ary predicates and \( n \) constants, there are \( p \cdot n^k \) instantiations.
Generalized Modus Ponens (GMP)

\[ p_1', p_2', \ldots, p_n', (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q) \quad \text{where } p'_i \theta = p_i \theta \text{ for all } i \]

\( q \theta \)

King(John), Greedy(y), (King(x) \land Greedy(x) \Rightarrow Evil(x))

- \( p_1' \) is \( King(John) \)
- \( p_1 \) is \( King(x) \)
- \( p_2' \) is \( Greedy(y) \)
- \( p_2 \) is \( Greedy(x) \)
- \( \theta \) is \{x/John,y/John\}
- \( q \) is \( Evil(x) \)
- \( q \theta \) is \( Evil(John) \)

- GMP used with KB of **definite clauses** (exactly one positive literal)

- All variables assumed universally quantified
Soundness of GMP

- Need to show that
  \[ p_1', \ldots, p_n', (p_1 \land \ldots \land p_n \Rightarrow q) \models q\theta \]
  provided that \( p_i'\theta = p_i\theta \) for all \( i \)

- Lemma: For any sentence \( p \), we have \( p \models p\theta \) by UI
  
  1. \( (p_1 \land \ldots \land p_n \Rightarrow q) \models (p_1 \land \ldots \land p_n \Rightarrow q)\theta = (p_1\theta \land \ldots \land p_n\theta \Rightarrow q\theta) \)
  2. \( p_1', \ldots, p_n' \models p_1' \land \ldots \land p_n' = p_1'\theta \land \ldots \land p_n'\theta \)
  3. From 1 and 2, \( q\theta \) follows by ordinary Modus Ponens
Unification

• We can get the inference immediately if we can find a substitution \( \theta \) such that \( \text{King}(x) \) and \( \text{Greedy}(x) \) match \( \text{King}(\text{John}) \) and \( \text{Greedy}(y) \)

\[ \theta = \{x/\text{John}, y/\text{John}\} \text{ works} \]

• \( \text{Unify}(\alpha, \beta) = \theta \) if \( \alpha\theta = \beta\theta \)
  – note: replace variables with terms!

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<tr>
<td>( \text{Knows}(\text{John},x) )</td>
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• **Standardizing apart** eliminates overlap of variables, e.g., \( \text{Knows}(z_{17},\text{OJ}) \)
Unification

- We can get the inference immediately if we can find a substitution \( \theta \) such that \( King(x) \) and \( Greedy(x) \) match \( King(John) \) and \( Greedy(y) \)

\[ \theta = \{ x/John, y/John \} \text{ works} \]

- Unify(\( \alpha, \beta \)) = \( \theta \) if \( \alpha\theta = \beta\theta \)

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<td>Knows(John,x)</td>
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<td>Knows(y,OJ)</td>
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\( \theta = \{x/\text{John}, y/\text{John}\} \) works

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- Standardizing apart eliminates overlap of variables, e.g., Knows(\(z_{17}\),OJ)
Unification

- We can get the inference immediately if we can find a substitution $\theta$ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

- $\text{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

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- **Standardizing apart** eliminates overlap of variables, e.g., Knows($z_{17}$,OJ)
Unification

- We can get the inference immediately if we can find a substitution $\theta$ such that \textit{King}(x) and \textit{Greedy}(x) match \textit{King}(\textit{John}) and \textit{Greedy}(\textit{y})

$\theta = \{x/\text{John}, y/\text{John}\}$ works

- \text{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

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</tr>
<tr>
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<td>\text{Knows}(x, \text{OJ})</td>
<td>$\emptyset$</td>
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- Standardizing apart eliminates overlap of variables, e.g., \text{Knows}(z_{17}, \text{OJ})
Unification

• To unify \( \text{Knows}(John,x) \) and \( \text{Knows}(y,z) \),
  \( \theta = \{y/John, x/z \} \) or \( \theta = \{y/John, x/John, z/John \} \)

• The first unifier is more general than the second.

• There is a single most general unifier (MGU) that is unique up to renaming of variables.
  \( \text{MGU} = \{ y/John, x/z \} \)
The unification algorithm

function Unify(x, y, \theta) returns a substitution to make x and y identical
inputs: x, a variable, constant, list, or compound
        y, a variable, constant, list, or compound
        \theta, the substitution built up so far

if \theta = failure then return failure
else if x = y then return \theta
else if VARIABLE?(x) then return Unify-Var(x, y, \theta)
else if VARIABLE?(y) then return Unify-Var(y, x, \theta)
else if COMPUND?(x) and COMPUND?(y) then
    return Unify ARGS[x], ARGS[y], Unify(OP[x], OP[y], \theta))
else if LIST?(x) and LIST?(y) then
    return Unify REST[x], REST[y], Unify(FIRST[x], FIRST[y], \theta))
else return failure
The unification algorithm

function UNIFY-VAR(var, x, θ) returns a substitution
    inputs: var, a variable
             x, any expression
             θ, the substitution built up so far
    if \{var/val\} ∈ θ then return UNIFY(val, x, θ)
    else if \{x/val\} ∈ θ then return UNIFY(var, val, θ)
    else if OCCUR-CHECK?(var, x) then return failure
    else return add \{var/x\} to θ
Unification

- Basic task: unify
  - $p_1, p_2, \ldots, p_n$
  - $q_1, q_2, \ldots, q_n$
- Proceed left to right, carry along current substitution $\theta$
- Compare $p_i$ with $q_i$,
  - predicates must match
  - apply existing substitution
  - unify instantiated pair, producing $\theta_i$
  - add new substitution to existing $\theta = \theta \cup \theta_i$
  - occurs check
Example knowledge base

• The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

• Prove that Col. West is a criminal
Example knowledge base, cont.

... it is a crime for an American to sell weapons to hostile nations:
\[
\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x,y,z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)
\]

Nono ... has some missiles, i.e., \(\exists x \ \text{Owns}(\text{Nono},x) \land \text{Missile}(x)\):
\[
\text{Owns}(\text{Nono},M_1) \text{ and } \text{Missile}(M_1)
\]

... all of its missiles were sold to it by Colonel West
\[
\text{Missile}(x) \land \text{Owns}(\text{Nono},x) \Rightarrow \text{Sells}(\text{West},x,\text{Nono})
\]

Missiles are weapons:
\[
\text{Missile}(x) \Rightarrow \text{Weapon}(x)
\]

An enemy of America counts as "hostile":
\[
\text{Enemy}(x,\text{America}) \Rightarrow \text{Hostile}(x)
\]

West, who is American ... \[
\text{American}(\text{West})
\]

The country Nono, an enemy of America ... \[
\text{Enemy}(\text{Nono},\text{America})
\]
function FOL-FC-Ask(\(KB, \alpha\)) returns a substitution or false

repeat until new is empty

\(new \leftarrow \{\}\)

for each sentence \(r\) in \(KB\) do

\((p_1 \land \ldots \land p_n \Rightarrow q) \leftarrow \text{STANDARDIZE-APART}(r)\)

for each \(\theta\) such that \((p_1 \land \ldots \land p_n)\theta = (p'_1 \land \ldots \land p'_n)\theta\)

for some \(p'_1, \ldots, p'_n\) in \(KB\)

\(q' \leftarrow \text{SUBST}(\theta, q)\)

if \(q'\) is not a renaming of a sentence already in \(KB\) or new then do

add \(q'\) to new

\(\phi \leftarrow \text{UNIFY}(q', \alpha)\)

if \(\phi\) is not fail then return \(\phi\)

add new to \(KB\)

return false
Forward chaining proof

American(West)  Missile(MI)  Owns(Nono,MI)  Enemy(Nono,America)
Forward chaining proof

\[
\text{Missile}(x) \land \text{Owns}(\text{Nono},x) \Rightarrow \text{Sells}(\text{West},x,\text{Nono})
\]

\[
\text{Missile}(x) \Rightarrow \text{Weapon}(x)
\]
Forward chaining proof

\[
\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x,y,z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)
\]
# Calendar

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Summary so far

- Reduction by propositionalization
  - Eliminate $\forall$ and $\exists$
  - With fn symbols infinitely many ground terms
  - Theorem: If a sentence $\alpha$ is entailed by an FOL KB, it is entailed by a finite subset of the propositionalized KB
  - Semi-decidable
  - Very slow in practice

- Generalized Modus Ponens
  - Inference with definite clauses
  - Replace instantiation step with unification
    \[
    q^\theta \quad \text{where } p_i^\theta = p_i^\theta \text{ for all } i
    \]

- $\text{UNIFY}(p,q)=\theta$ where $\text{SUBST}(\theta,p)=\text{SUBST}(\theta,q)$
- Forward chaining in FOL: lifted version of FC in PL
Forward chaining proof

- American(x) \land Weapon(y) \land Sells(x,y,z) \land Hostile(z) \Rightarrow Criminal(x)
- Missile(x) \land Owns(Nono,x) \Rightarrow Sells(West,x,Nono)
- Missile(x) \Rightarrow Weapon(x)
- Enemy(x,America) \Rightarrow Hostile(x)
- Owns(Nono,M1) and Missile(M1)
- American(West)
- Enemy(Nono,America)
Properties of forward chaining

• Forward chaining is widely used in deductive databases

• Sound and complete for first-order definite clauses
• May not terminate in general if $\alpha$ is not entailed
• This is unavoidable: entailment with definite clauses is semidecidable

• **Datalog** = first-order definite clauses + no functions
  - FC terminates for Datalog in finite number of iterations $(p \cdot n^k)$ ground terms
Matching facts against rules:

Hard matching example

- **Colorable()** is inferred iff the CSP has a solution
- CSPs include 3SAT as a special case, hence matching is NP-hard
- Query complexity vs. data complexity

\[
\text{Diff}(wa,nt) \land \text{Diff}(wa,sa) \land \text{Diff}(nt,q) \land \\
\text{Diff}(nt,sa) \land \text{Diff}(q,nsw) \land \text{Diff}(q,sa) \land \\
\text{Diff}(nsw,v) \land \text{Diff}(nsw,sa) \land \text{Diff}(v,sa) \Rightarrow \\
\text{Colorable()}
\]

\[
\text{Diff}(Red,Blue) \quad \text{Diff} (Red,Green) \\
\text{Diff}(Green,Red) \quad \text{Diff}(Green,Blue) \\
\text{Diff}(Blue,Red) \quad \text{Diff}(Blue,Green)
\]
Efficiency of forward chaining

\[
p_1', p_2', \ldots, p_n', (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q)_{q\theta} \quad \text{where } p_i'\theta = p_i \theta \text{ for all } i
\]

- Pattern matching itself can be expensive:
  - Use indexing to unify sentences that have a chance of unifying
    - Knows(x,y) vs Brother(u,v)
  - Database indexing allows O(1) retrieval of known facts
  - e.g., query Missile(x) retrieves Missile(M_1)
Efficiency of forward chaining

\[ \frac{p_1', p_2', \ldots, p_n'}{q'} \quad (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q') \quad \text{where } p_i'\theta = p_i \theta \text{ for all } i \]

- Matching rules against known facts

  Conjunct ordering problem
  Missile(x) \land Owns(Nono,x) \Rightarrow Sells(West, x, Nono)

  NP-hard in general, but can use heuristics used for CSPs

  Rule-matching tractable when CSP is tractable
Efficiency of forward chaining

1. Incremental forward chaining: no need to match a rule on iteration $k$ if a premise wasn't added on iteration $k-1$
   \[ \Rightarrow \text{match each rule whose premise contains a newly added positive literal} \]
2. Retain partial matches and complete them incrementally as new facts arrive
Efficiency of forward chaining

Forward chaining infers everything, most of which can be irrelevant to the goal

- Solution: allow only those bindings that are relevant to the goal
  - Use generic backward chaining
- Add Magic(x) extra conjunct to rules and Magic(c) to the KB
  - E.g. Magic(West)
Backward chaining example
Backward chaining example
Backward chaining example

- Criminal(West)
- American(West)
- Weapon(y)
- Sells(x,y,z)
- Hostile(z)
- \{x/West\}
- \{\}

Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining algorithm

function FOL-BC-Ask(KB, goals, \theta) returns a set of substitutions
  inputs: KB, a knowledge base
  goals, a list of conjuncts forming a query
  \theta, the current substitution, initially the empty substitution \{\}
  local variables: ans, a set of substitutions, initially empty
  if goals is empty then return \{\theta\}
  q' \leftarrow \text{SUBST}(\theta, \text{FIRST}(goals))
  for each r in KB where \text{STANDARDIZE-APART}(r) = (p_1 \land \ldots \land p_n \Rightarrow q)
    and \theta' \leftarrow \text{UNIFY}(q, q') succeeds
    ans \leftarrow \text{FOL-BC-Ask}(KB, [p_1, \ldots, p_n|\text{REST}(goals)], \text{COMPOSE}(\theta, \theta')) \cup ans
  return ans

\text{SUBST}(\text{COMPOSE}(\theta_1, \theta_2), p) = \text{SUBST}(\theta_2, \text{SUBST}(\theta_1, p))
Properties of backward chaining

• Depth-first recursive proof search: space is linear in size of proof
  – But not in size of data (bindings)
• Incomplete due to infinite loops
  – fix by checking current goal against every goal on stack
• Inefficient due to repeated subgoals (both success and failure)
  – fix using caching of previous results (extra space)
• Widely used for logic programming (Prolog)
• Appending two lists to produce a third:

\[
\text{append}([], Y, Y).
\]
\[
\text{append}([X|L], Y, [X|Z]) :- \text{append} (L, Y, Z).
\]

• query: \text{append} (A, B, [1,2]) ?

• answers: \begin{align*}
A & = [] & B & = [1,2] \\
A & = [1,2] & B & = []
\end{align*}
Logic programming: Prolog

- Algorithm = Logic + Control

- Basis: backward chaining with Horn clauses + bells & whistles
Widely used in Europe, Japan (basis of 5th Generation project)
Compilation techniques ⇒ 60 million LIPS

- Program = set of clauses = head :- literal₁, ..., literalₙ.
  criminal(X) :- american(X), weapon(Y), sells(X,Y,Z), hostile(Z).

- Depth-first, left-to-right (within rule), top-down (within rule-set) backward chaining
- Built-in predicates for arithmetic etc., e.g., X is Y*Z+3
- Built-in predicates that have side effects (e.g., input and output predicates, assert/retract predicates)
- No occurs-check in unification – may produce results not entailed
- No checks for infinite loops – incomplete even for definite clauses
- Prolog: no caching; Tabled Logic Programming: memoization
- Database semantics:
  - Unique names assumption
  - Closed-world assumption ("negation as failure")
    - e.g., given alive(X) :- not dead(X).
    - alive(joe) succeeds if dead(joe) fails
  - Closed domain assumption
Resolution: brief summary

- Full first-order version:

\[
\begin{align*}
\ell_1 \lor \cdots \lor \ell_k & \quad m_1 \lor \cdots \lor m_n \\
\hline
(\ell_1 \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n)\theta
\end{align*}
\]

where \(\text{Unify}(\ell_i, \neg m_j) = \theta\).

- The two clauses are assumed to be standardized apart so that they share no variables.

- For example,

\[
\begin{align*}
\neg Rich(x) \lor Unhappy(x) & \quad Rich(Ken) \\
\hline
Unhappy(Ken)
\end{align*}
\]

with \(\theta = \{x/Ken\}\)

- Apply resolution steps to \(\text{CNF}(KB \land \neg \alpha)\); complete (with factoring) for FOL
Conversion to CNF

• Everyone who loves all animals is loved by someone:
  \[ \forall x [\forall y \text{Animal}(y) \Rightarrow \text{Loves}(x,y)] \Rightarrow [\exists y \text{Loves}(y,x)] \]

• 1. Eliminate biconditionals and implications

  \[
  \begin{align*}
  A \iff B & \text{ becomes } (A \Rightarrow B) \land (B \Rightarrow A) \\
  A \Rightarrow B & \text{ becomes } \neg A \lor B
  \end{align*}
  \]

  \[
  \begin{align*}
  \forall x [\forall y \neg \text{Animal}(y) \lor \text{Loves}(x,y)] & \lor [\exists y \text{Loves}(y,x)] \\
  \forall x [\forall y \neg \text{Animal}(y) \lor \text{Loves}(x,y)] & \lor [\exists y \text{Loves}(y,x)]
  \end{align*}
  \]

• 2. Move \( \neg \) inwards: \( \neg \forall x \ p \equiv \exists x \ \neg p, \ \neg \exists x \ p \equiv \forall x \ \neg p \)

  \[
  \begin{align*}
  \forall x [\exists y \neg (\neg \text{Animal}(y) \lor \text{Loves}(x,y))] & \Rightarrow [\exists y \text{Loves}(y,x)] \\
  \forall x [\exists y \neg \text{Animal}(y) \land \neg \text{Loves}(x,y)] & \lor [\exists y \text{Loves}(y,x)] \\
  \forall x [\exists y \text{Animal}(y) \land \neg \text{Loves}(x,y)] & \lor [\exists y \text{Loves}(y,x)]
  \end{align*}
  \]
Conversion to CNF contd.

• 3. Standardize variables: each quantifier should use a different one

\[\forall x \left( \exists y \ Animal(y) \land \neg Loves(x,y) \right) \lor \left( \exists z \ Loves(z,x) \right)\]

• 4. Skolemize: a more general form of existential instantiation.

Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables:

\[\forall x \left( Animal(F(x)) \land \neg Loves(x,F(x)) \right) \lor Loves(G(x),x)\]

• 5. Drop universal quantifiers:

\[\left( Animal(F(x)) \land \neg Loves(x,F(x)) \right) \lor Loves(G(x),x)\]

• 6. Distribute \(\lor\) over \(\land\):

\[\left( Animal(F(x)) \lor Loves(G(x),x) \right) \land \left( \neg Loves(x,F(x)) \lor Loves(G(x),x) \right)\]
... it is a crime for an American to sell weapons to hostile nations:

\[ \text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x,y,z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x) \]

Nono … has some missiles, i.e., \( \exists x \ \text{Owns}(\text{Nono},x) \land \text{Missile}(x) \):

\[ \text{Owns}(\text{Nono},M_1) \land \text{Missile}(M_1) \]

… all of its missiles were sold to it by Colonel West

\[ \text{Missile}(x) \land \text{Owns}(\text{Nono},x) \Rightarrow \text{Sells}(\text{West},x,\text{Nono}) \]

Missiles are weapons:

\[ \text{Missile}(x) \Rightarrow \text{Weapon}(x) \]

An enemy of America counts as "hostile“:

\[ \text{Enemy}(x,\text{America}) \Rightarrow \text{Hostile}(x) \]

West, who is American …

\[ \text{American}(\text{West}) \]

The country Nono, an enemy of America …

\[ \text{Enemy}(\text{Nono},\text{America}) \]
Resolution proof: definite clauses

\[ \neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x,y,z) \lor \neg Hostile(z) \lor Criminal(x) \]

\[ \neg Criminal(West) \]

\[ \neg American(West) \lor \neg Weapon(y) \lor \neg Sells(West,y,z) \lor \neg Hostile(z) \]

\[ American(West) \]

\[ \neg Missile(x) \lor Weapon(x) \]

\[ \neg Missile(x) \lor \neg Owns(Nono,x) \lor Sells(West,x,Nono) \]

\[ Missile(M1) \]

\[ \neg Missile(x) \lor \neg Owns(Nono,x) \lor Sells(West,x,Nono) \]

\[ Missle(M1) \]

\[ \neg Sells(West,M1,z) \lor \neg Hostile(z) \]

\[ \neg Missile(y) \lor \neg Sells(West,y,z) \lor \neg Hostile(z) \]

\[ \neg Missile(y) \lor \neg Sells(West,y,z) \lor \neg Hostile(z) \]

\[ Missile(M1) \]

\[ \neg Missile(M1) \lor \neg Owns(Nono,M1) \lor \neg Hostile(Nono) \]

\[ Missle(M1) \]

\[ \neg Owns(Nono,M1) \lor \neg Hostile(Nono) \]

\[ \neg Owns(Nono,M1) \lor \neg Hostile(Nono) \]

\[ \neg Enemy(x,America) \lor Hostile(x) \]

\[ \neg Enemy(x,America) \lor Hostile(x) \]

\[ \neg Enemy(Nono,America) \]

\[ \neg Enemy(Nono,America) \]

\[ \neg Hostile(Nono) \]

\[ \neg Hostile(Nono) \]
Efficient Resolution

• Resolution proofs can be long

• Strategies:
  – Unit Preference
  – Set of support
  – Input resolution
    • Complete for Horn clauses
  – Linear Resolution
    • Complete in general
Converting to clause form (Try this example)

\( \forall x, y \; P(x) \land P(y) \land I(x,27) \land I(y,28) \rightarrow S(x, y) \)

\( P(A), P(B) \)

\( I(A,27) \lor I(A,28) \)

\( I(B,27) \)

\( \neg S(B, A) \)

Prove \( I(A,27) \)
Example: Resolution
Refutation Prove  $I(A,27)$
Example: Answer Extraction

\neg I(A, u) \lor \text{Ans}(u)
\text{negation of wff to be proved with answer literal)

I(A, 28) \lor \text{Ans}(27)

\neg P(x) \lor \neg P(y) \lor \neg I(x, 27) \lor \neg I(y, 28) \lor S(x, y)

P(A)

\{A/y\}

P(B)

\neg P(x) \lor \neg I(x, 27) \lor S(x, A) \lor \text{Ans}(27)

I(B, 27)

\neg I(B, 27) \lor S(B, A) \lor \text{Ans}(27)

\{B/x\}

\neg S(B, A)

S(B, A) \lor \text{Ans}(27)

\text{Ans}(27)
Unit resolution is (refutation) complete for Horn clauses : example

(Non-unit) resolution is complete!

Can transform any non-unit resolution proof to unit resolution proof
Unit resolution is (refutation) complete for Horn clauses: general case

\[ \neg S_1, \neg S_2, B \]

\[ \neg A, \neg S_1, B \]

\[ A, \neg S_2 \]

Only unit resolution here. \( S_2 \) eliminated one at a time.