Stat120B: Introduction to Prabability and Statistics **Lecture Notes Archive**

Moment Generating Function

Lecturer: Gang Liang

1 Definition

Definition 1 *For a random variable X, its moment generating fuction (MGF) is defined as*

$$M_X(t) = \mathrm{E}\left(\exp(tX)\right).$$

More specifically, we have

$$M_X(t) = \begin{cases} \int \exp(tx) f(x) \mathrm{d}x & \textit{continous rv's} \\ \sum_x \exp(tx) p(x) & \textit{discrete rv's.} \end{cases}$$

2 Some useful properties

1. $M_X(0) = 1$;
2. $M_X(t)$ may not exist for $t \neq 0$;
3. If $M_X(t)$ exists at an interval around 0, then we have
$$M_X(t) \Longleftrightarrow f_X(x),$$
i.e., the moment generating function uniquely determines the density function. That is knowing a MGF of a rv is equavilent to knowing its density function;
4. If $Y = a + bX$, where a, b are constants, then easy to show
$$M_Y(t) = \exp(at) M_X(t);$$
5. If $Z = X + Y$ and X, Y are independent, then
$$M_Z(t) = M_X(t) M_Y(t).$$
This is a very useful property of the MGFs;
6. The definition of the MGF can be easily extended to the multivariate cases. For instanct, let $Z = (X, Y)$, then
$$M_Z(s, t) = \mathrm{E}(\exp(sX + tY));$$
7. The last one is the most important property of a moment generating function. Suppose we would like to do Taylor expansion of a MGF around 0. We already know $M_X(0) = 1$, then let us consider
$$\begin{aligned} M'_X(0) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\int \exp(tx) f(x) \mathrm{d}x \right) \Big|_{t=0} = \left(\int \frac{\mathrm{d}}{\mathrm{d}t} \exp(tx) f(x) \mathrm{d}x \right) \Big|_{t=0} \\ &= \left(\int x \exp(tx) f(x) \mathrm{d}x \right) \Big|_{t=0} \\ &= \int x f(x) \mathrm{d}x \\ &= \mathrm{E}(X). \end{aligned}$$

In general,

$$M_X^{(r)}(0) = \left(\int \frac{\mathrm{d}^r}{\mathrm{d}t^r} exp(tx) f(x) \mathrm{d}x \right) \Big|_{t=0} = \int x^r f(x) \mathrm{d}x = \mathrm{E}(X^r).$$

It is why $M_X(t)$ is called the moment generating function of X.

3 Examples

3.1 Poisson

Let X be a Poisson random variable, then

$$P(X = k) = \frac{\lambda^k}{k!} \exp(-\lambda).$$

$$M(x) = \sum_{k=0}^{\infty} \exp(tk) \frac{\lambda^k}{k!} \exp(-\lambda) = \exp(-\lambda) \sum_{k=0}^{\infty} \frac{(\lambda \exp(t))^k}{k!} = \exp(\lambda \exp(t) - \lambda).$$

So we have

$$\mathrm{E}(X) = M'(0) = \lambda,$$

and

$$\mathrm{E}(X^2) = M''(0) = \lambda^2 + \lambda.$$

So $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}X)^2 = \lambda$.

3.2 Gamma

Let X be a Gamma random variable: $X \sim \Gamma(\alpha, \lambda)$. Its density function is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x), x > 0.$$

Its moment generating function is

$$\begin{aligned}
M(t) &= \int_0^\infty \exp(tx) \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x) \mathrm{d}x = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \exp(-(\lambda - t)x) \mathrm{d}x \\
&= \frac{\lambda^\alpha}{(\lambda - t)^\alpha \Gamma(\alpha)} \int_0^\infty [(\lambda - t)x]^{\alpha-1} \exp(-(\lambda - t)x) \mathrm{d}((\lambda - t)x) \\
&= \frac{\lambda^\alpha}{(\lambda - t)^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} \exp(-y) \mathrm{d}y \quad \text{Let } y = (\lambda - t)x \\
&= \left(\frac{\lambda}{\lambda - t} \right)^\alpha
\end{aligned}$$

mean, $E(X^2)$ and variance: α/λ, $\alpha(\alpha + 1)/\lambda^2$.

3.3 Exercise

Compute the MGF for normal distributions.