Epipolar Geometry

CS 211A

Degrees of Freedom(DoF)

- Number of parameters than can be changed in the matrix representing the transformation
- Consider a 3x3 matrix for 2D translation and rotation
  - 2 translation
  - 1 rotation
- 3 degrees of freedom

Affected elements

- Number of matrix elements affected can be greater than the DoF
- Rotation + Translation 3x3 matrix
  - 6 entries are affected
  - DoF is 3
- DoF <= Elements Affected
- Add Scaling and Shear
  - Is it 7 DoF?
  - No, DoF is still 6

Why?

- Consider x' = ax+by
  - Scaling parameter is a
  - Shear is b
  - But they can be considered similar to sine and cosine of rotation
  - Scaling, shear and rotational degree of freedom affect same elements of the matrix
- Any added constraint on the matrix elements
  - Reduction in DoF

Why 8 elements in H and F?

Matrix Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Equations:

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

There are 9 numbers $h_{11},\ldots,h_{33}$, so are there 9 DOF?

No. Note that we can multiply all h_{ij} by nonzero k without changing the equations:

$$x' = \frac{kh_{11}x + kh_{12}y + kh_{13}}{kh_{31}x + kh_{32}y + kh_{33}} \qquad\Rightarrow\qquad x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{kh_{21}x + kh_{22}y + kh_{23}}{kh_{31}x + kh_{32}y + kh_{33}} \qquad\Rightarrow\qquad y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Degrees of Freedom

- 9 numbers but their ratios are important
- Removes one degree of freedom
- 8 degrees of freedom

Force 8 degrees of freedom

One approach: Set $h_{33} = 1$.

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Second approach: Impose unit vector constraint

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Subject to the constraint: $h_{11}^2 + h_{12}^2 + h_{13}^2 + h_{21}^2 + h_{22}^2 + h_{23}^2 + h_{31}^2 + h_{32}^2 + h_{33}^2 = 1$

First Approach

Setting $h_{33} = 1$

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + 1)x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + 1)y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' = x'$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' = y'$$

First Approach

2N x 8 | 8 x 1 | 2N x 1

Point 1, Point 2, Point 3, Point 4

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4 \\ & & & \vdots & & & & \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \\ \vdots \end{bmatrix}$$

additional points

First Approach

Linear equations

$$\underset{2N\times 8}{A}\ \underset{8\times 1}{h} = \underset{2N\times 1}{b}$$

Solve:

$$\underset{8\times 2N}{A^T}\ \underset{2N\times 8}{A}\ \underset{8\times 1}{h} = \underset{8\times 2N}{A^T}\ \underset{2N\times 1}{b}$$

$$\overbrace{(A^T\ A)}^{8\times 8}\ \underset{}{\overset{8\times 1}{h}} = \overbrace{(A^T\ b)}^{8\times 1}$$

$$h = (A^T\ A)^{-1}\ (A^T\ b)$$

Matlab: $h = A \backslash b$

Limitation

- Poor Conditioning of Matrices Involved
- Sensitive to noise
- If element (3,3) is actually 0 we cannot get the right answer

Second Approach: Norm=1

||h|| = 1

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + h_{33})x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + h_{33})y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - h_{33}x' = 0$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - h_{33}y' = 0$$

Second Approach

2N x 9 · **9 x 1** · **2N x 1**

4 POINTS

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 & -y'_1 \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \\ \end{bmatrix}$$

additional points $\vdots$ $\vdots$

Second Approach

Homogeneous equations

$$\underset{2N\times 9}{A}\;\underset{9\times 1}{h} = \underset{2N\times 1}{0}$$

Solve:

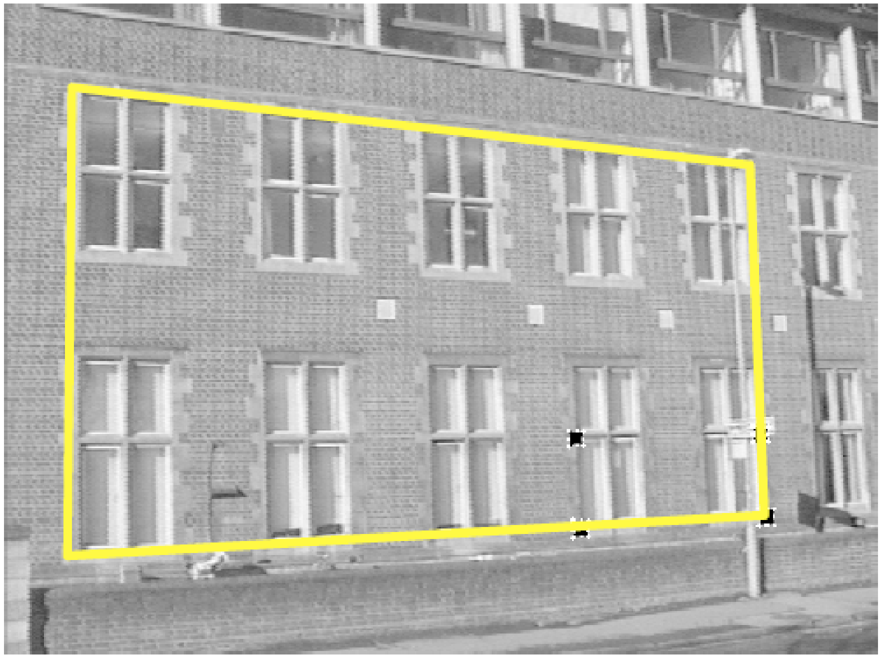
$$\underset{9\times 2N}{A^T}\;\underset{2N\times 9}{A}\;\underset{9\times 1}{h} = \underset{9\times 2N}{A^T}\;\underset{2N\times 1}{0}$$

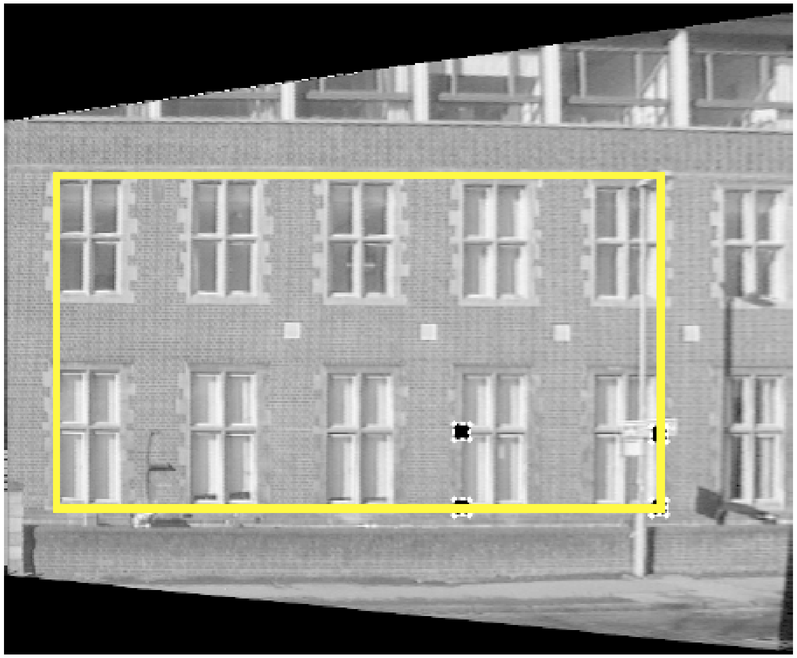
$$\overbrace{(A^T\;A)}^{9\times 9}\;\underset{9\times 1}{h} = \underset{9\times 1}{0}$$

SVD of $A^TA = U\;D\;U^T$

Let h be the column of U (unit eigenvector) associated with the smallest eigenvalue in D. (if only 4 points, that eigenvalue will be 0)

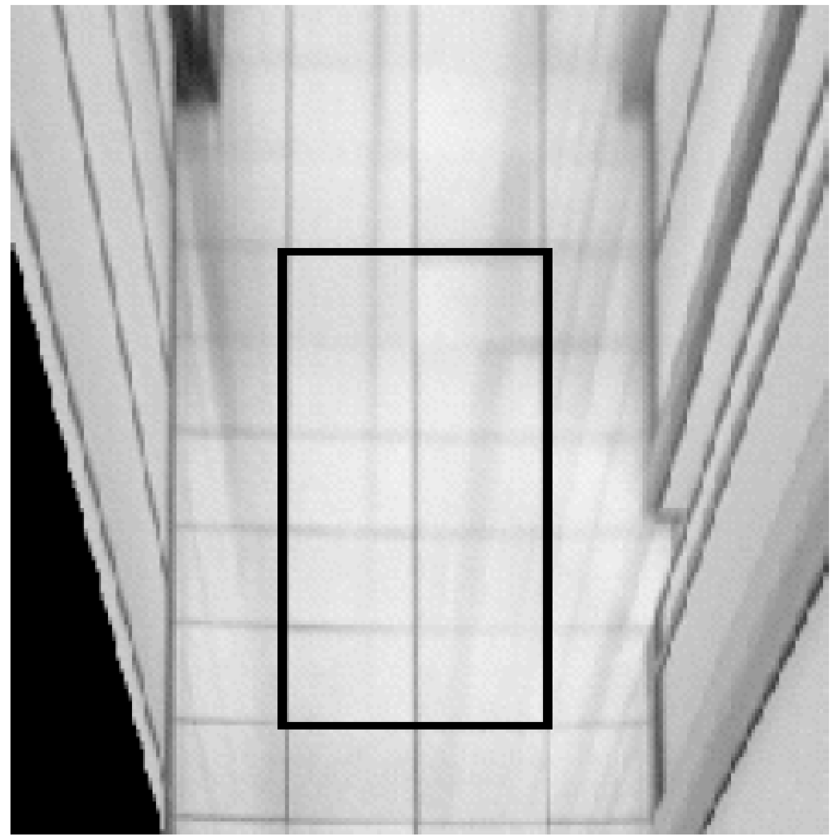
Removing Perspective Effects





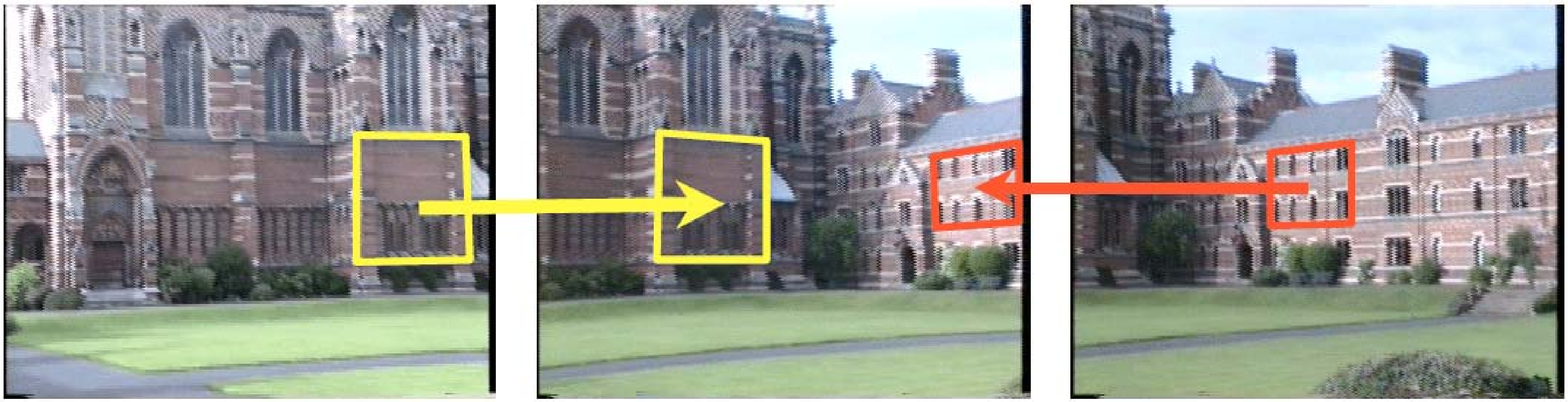
Hartley and Zisserman

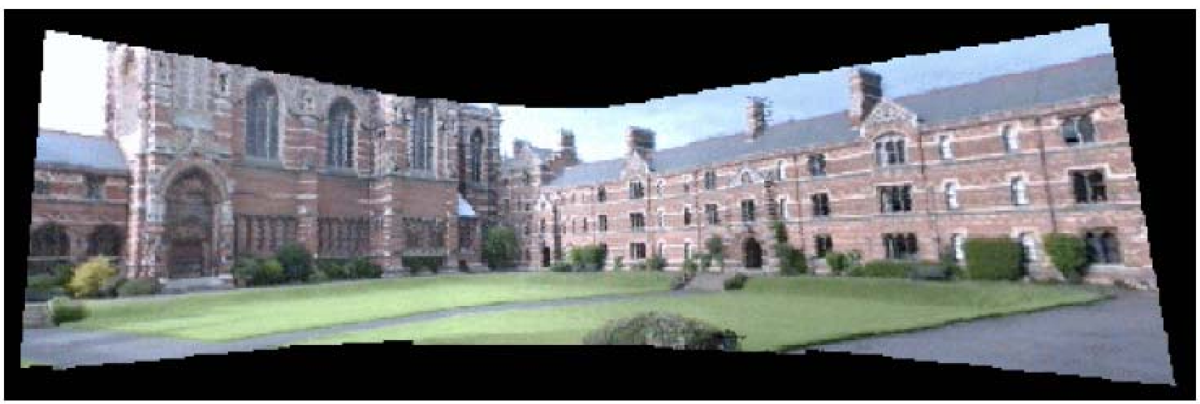
Bird's Eye View



Hartley and Zisserman

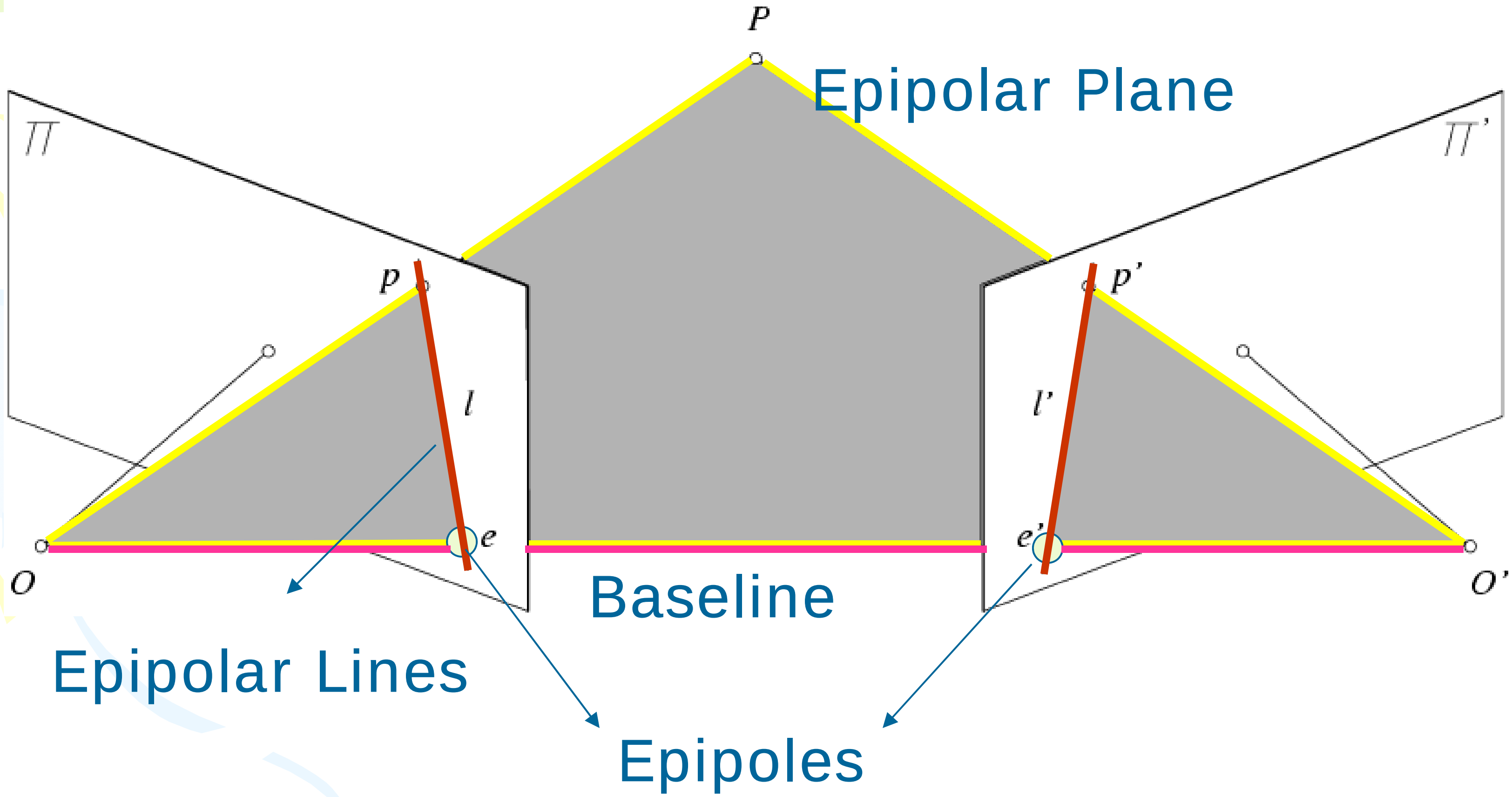
Mosaicing





from Hartley & Zisserman

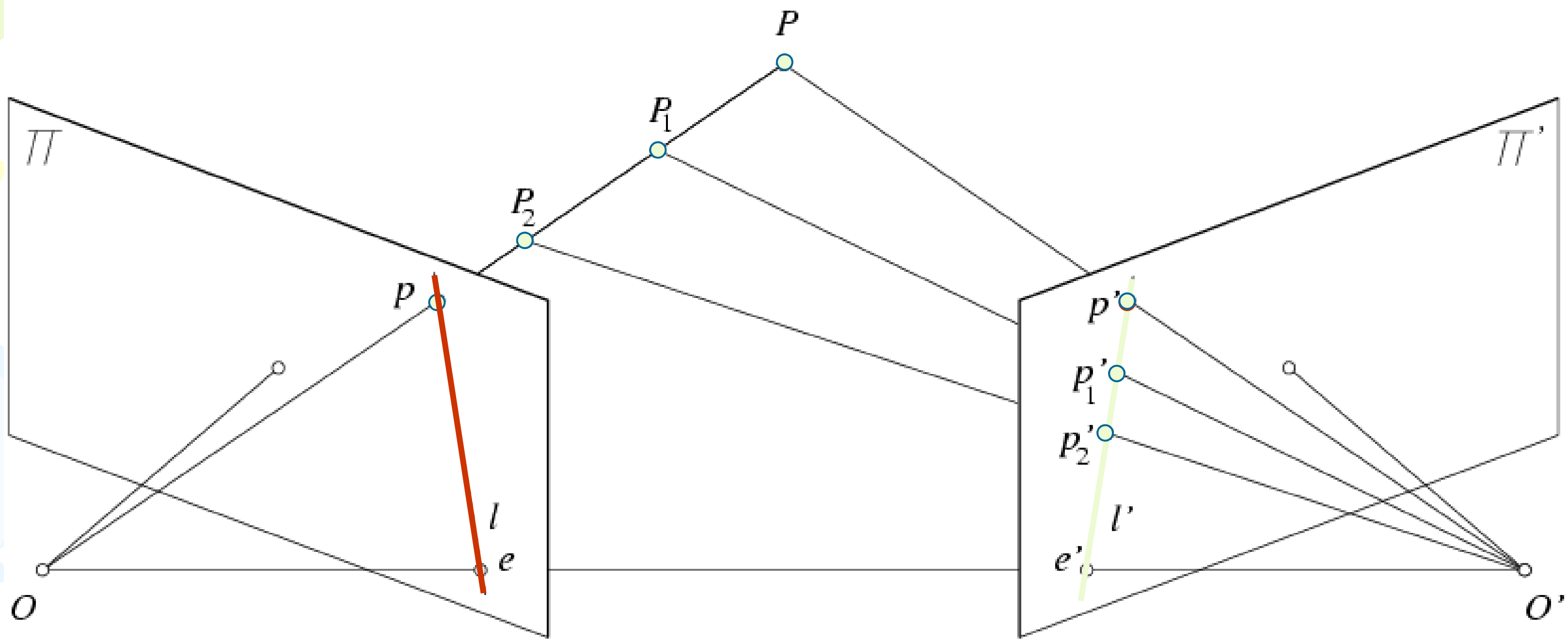
Epipolar Geometry



Epipolar geometry

- Epipolar planes are a pencil of planes rooted at the baseline
- Baseline connects the two COPs
- A point in one image maps to a epipolar line in another image
  - Reduces search space for correspondence from 2D plane to 1D line
- All epipolar lines pass through the epipole of the image

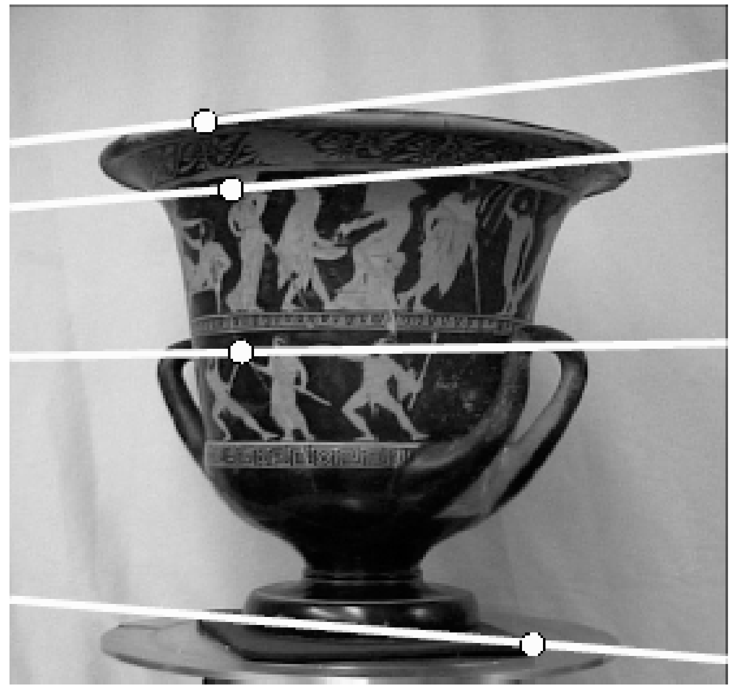
Epipolar Constraint



- Potential matches for *p* have to lie on the corresponding epipolar line *l'*.

- Potential matches for *p'* have to lie on the corresponding epipolar line *l*.

Epipolar lines

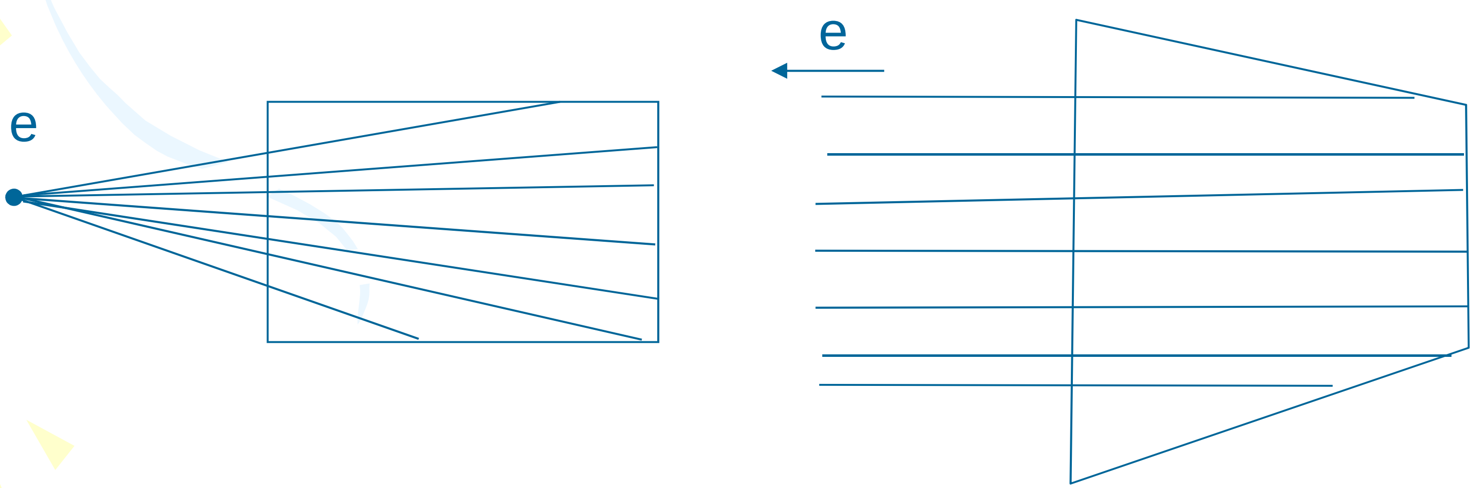


Degrees of Freedom

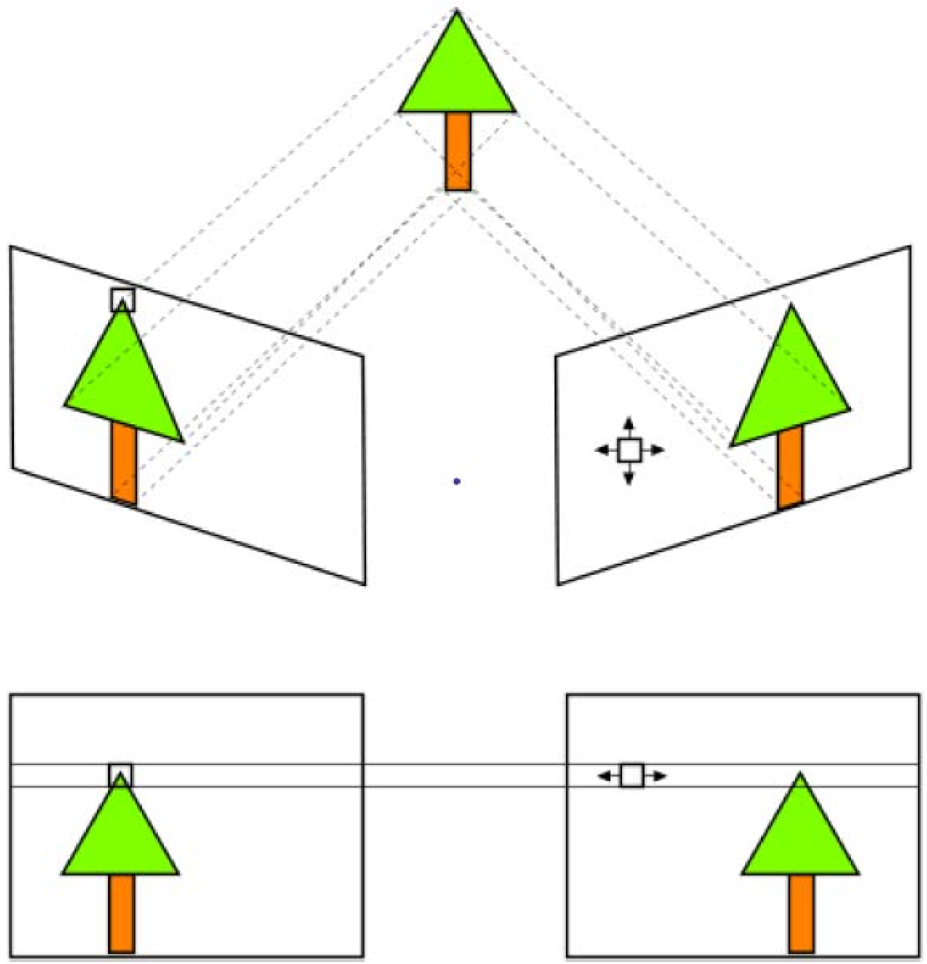
- Same as homography
- In addition, det(F) = 0
  - Removes one more degree of freedom
- 7 degrees of freedom
- Essential Matrix
  - 5 degrees of freedom

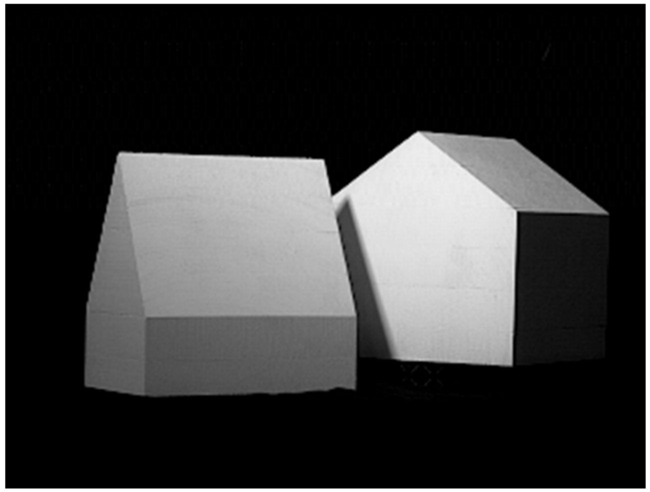
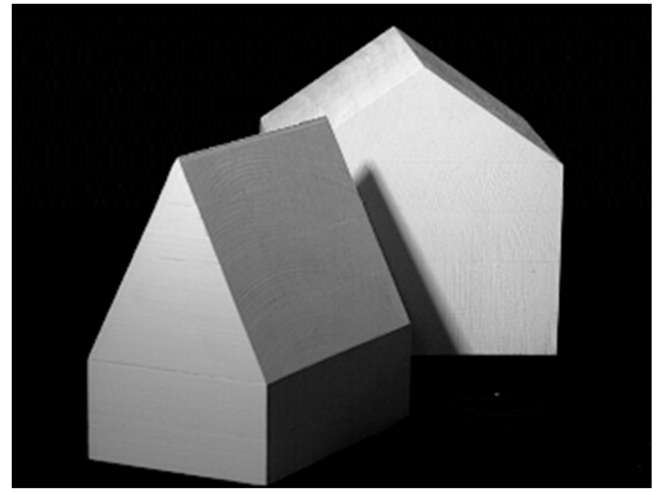
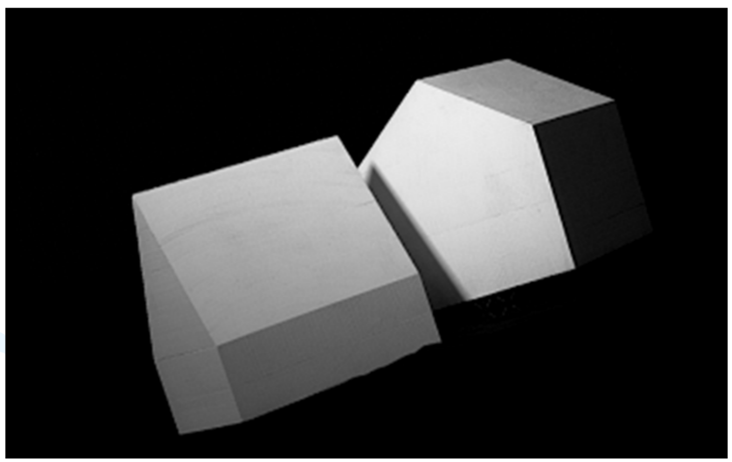
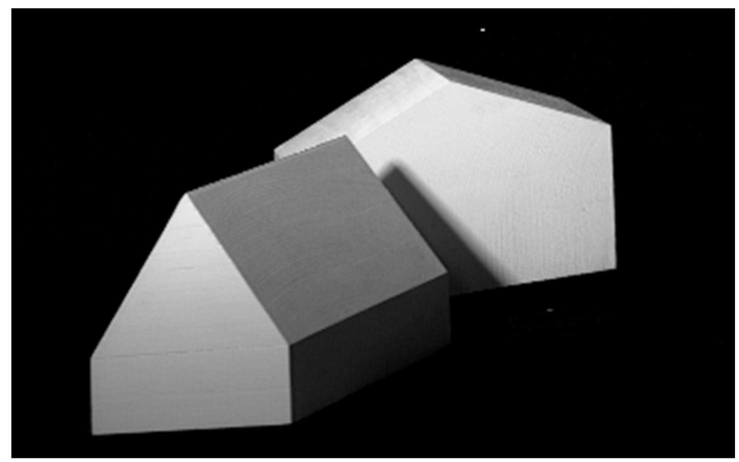
Rectification

- $\mathbf{E} = \mathbf{R}[\mathbf{t}]_{\mathbf{x}}$
- **R** and **t** are the rotation and translation required to make the image planes parallel and away by a a pure translation
- Called rectification



Rectification

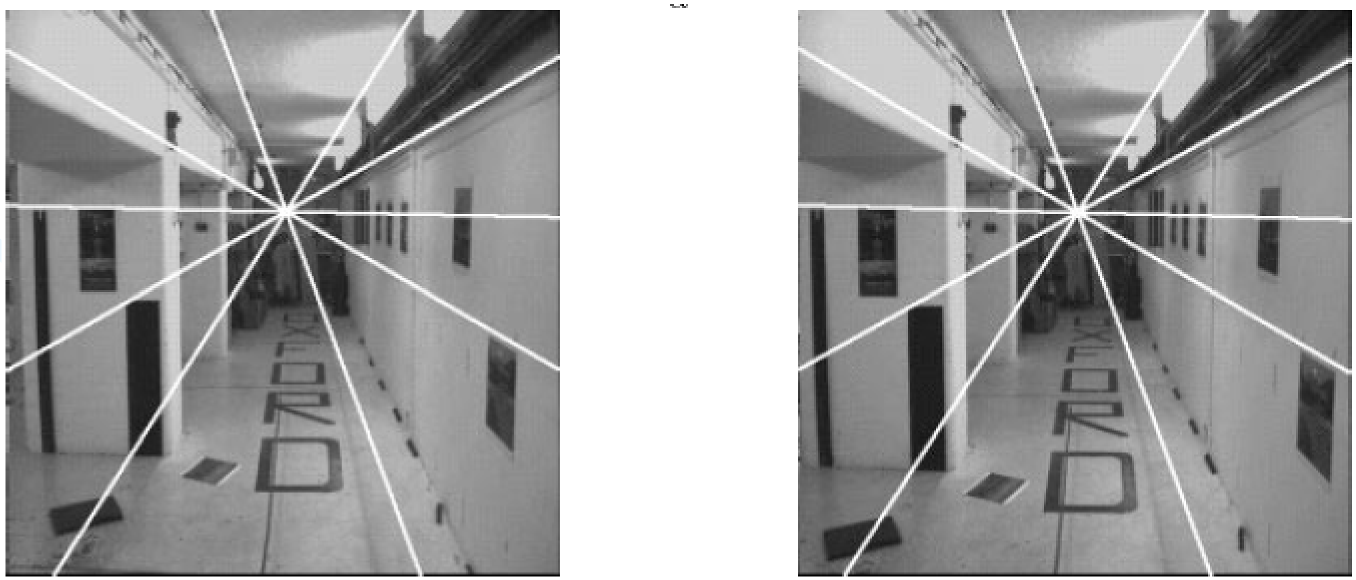


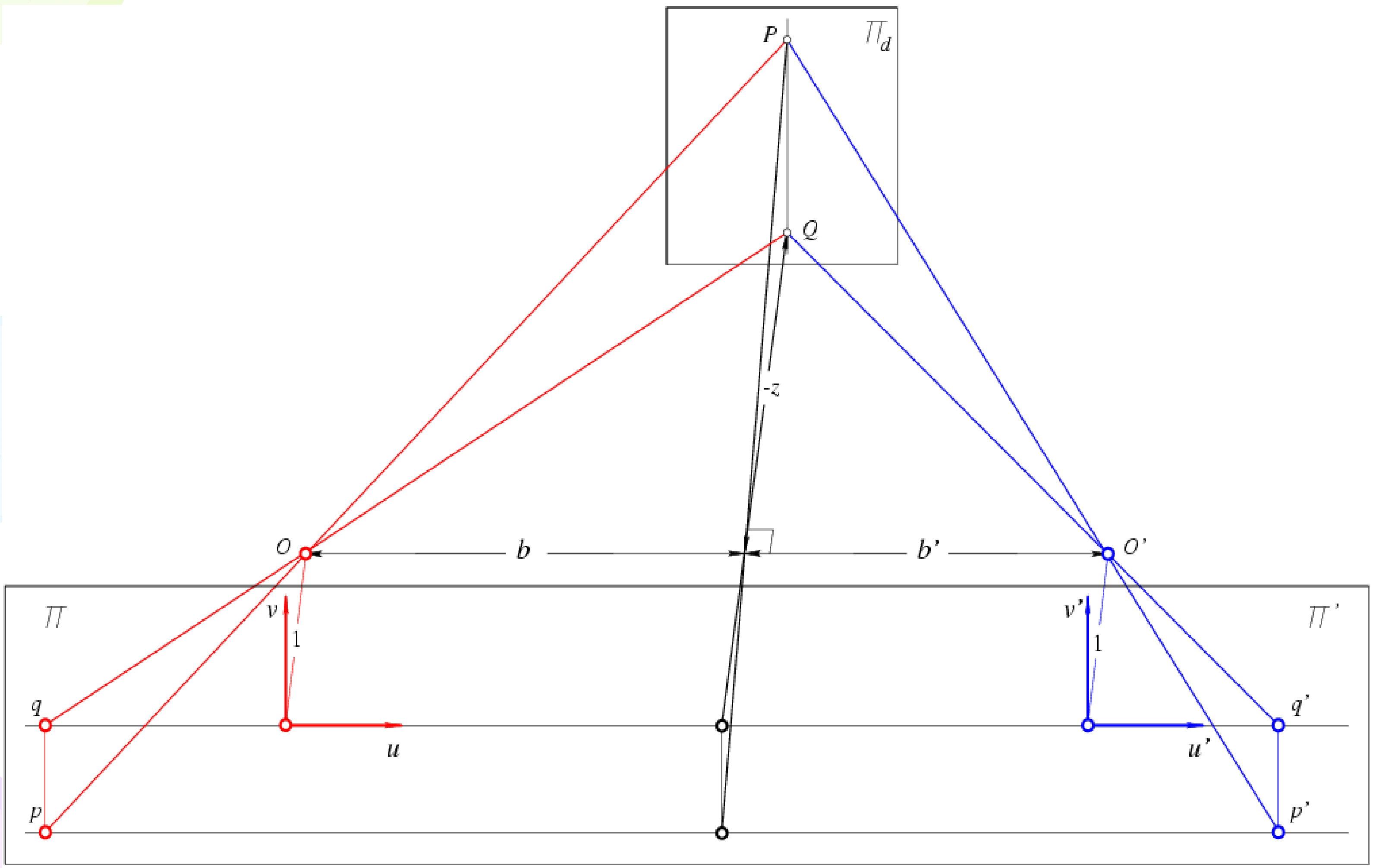



Optical Flow

- p’ = p + Kt/Z



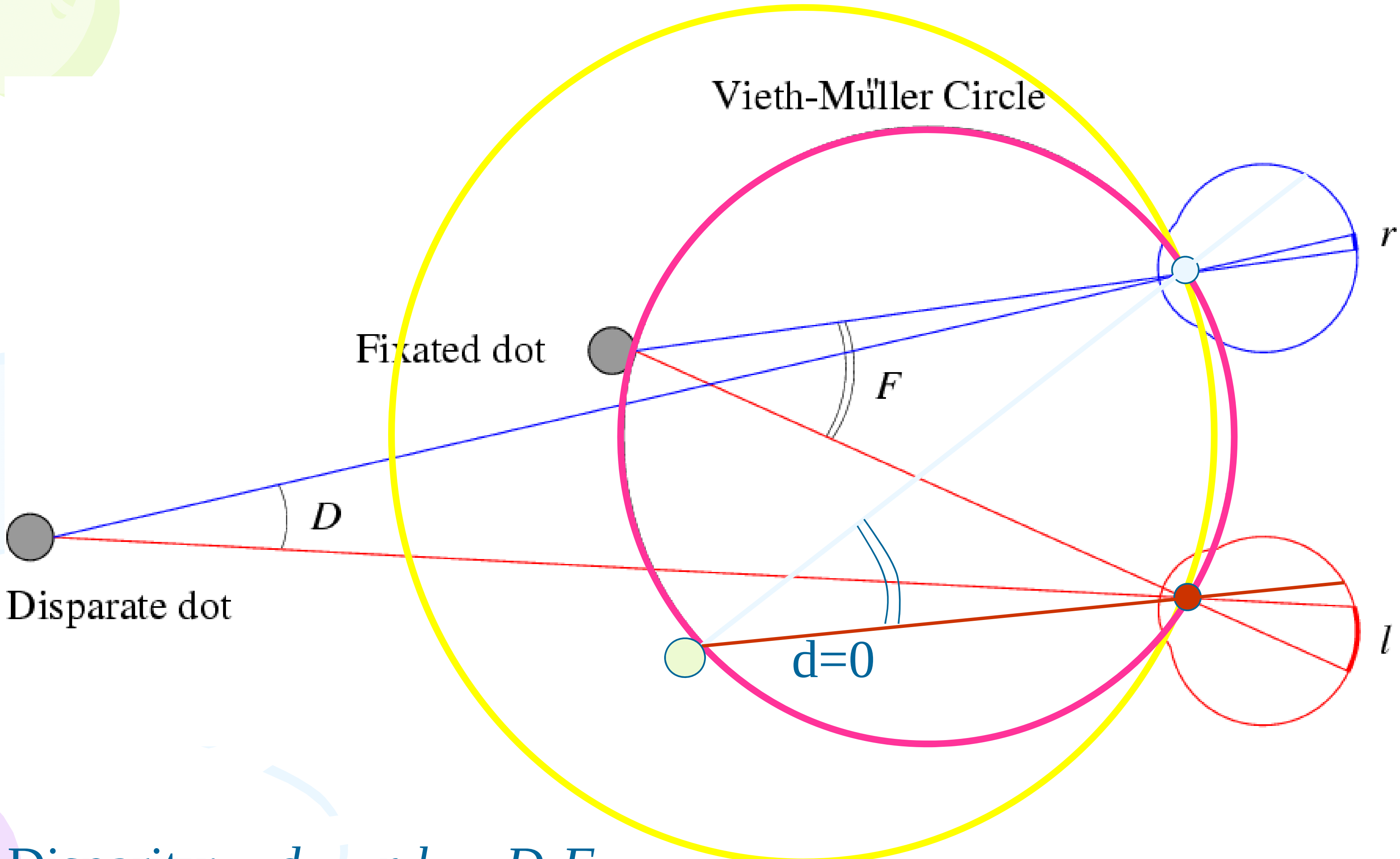
Reconstruction from Rectified Images



Disparity: $d=u'-u$.

Depth: $z = -B/d$.

Human Stereopsis: Reconstruction

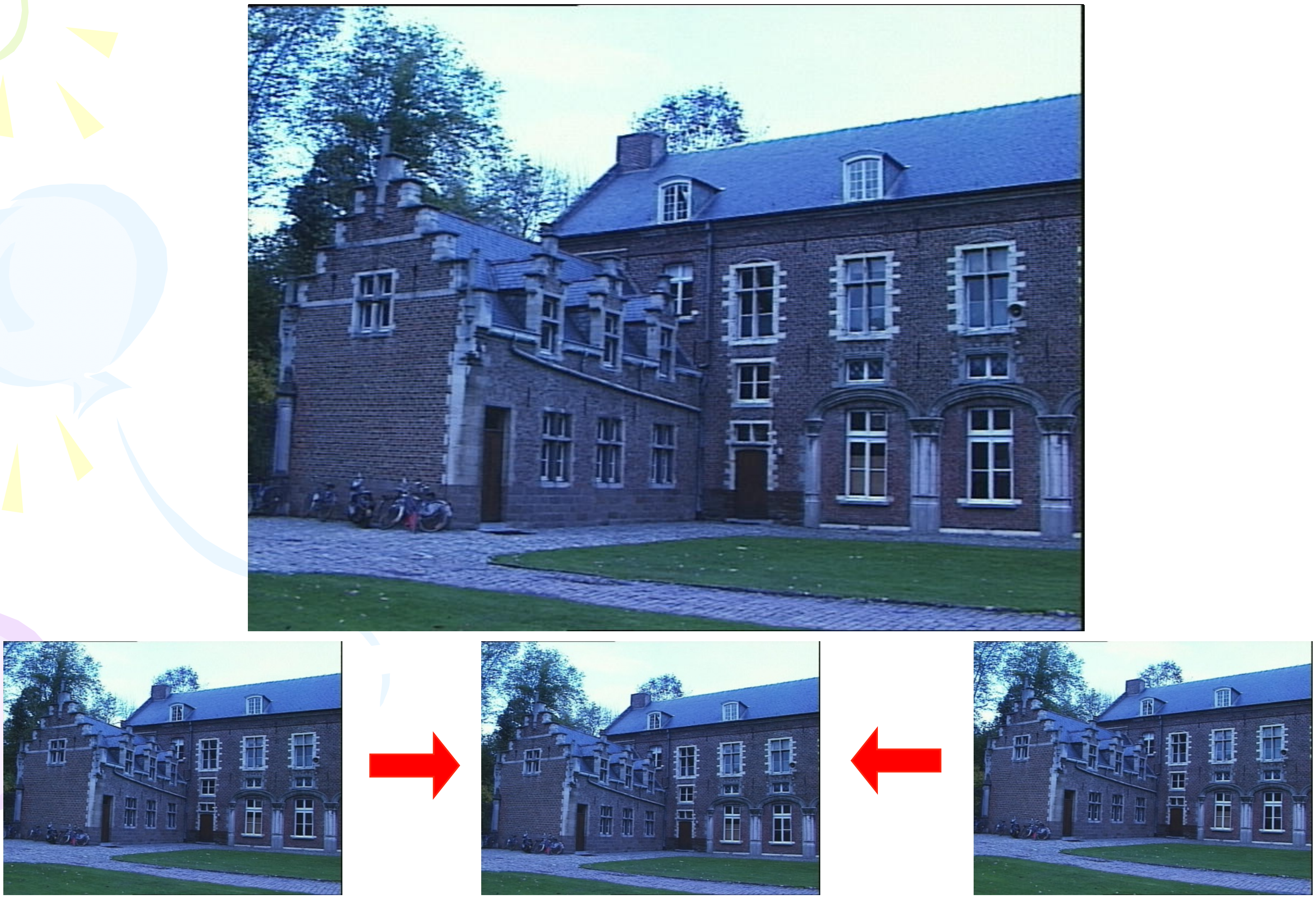


d<0

Disparity: $d = r\text{-}l = D\text{-}F.$

In 3D, the horopter.

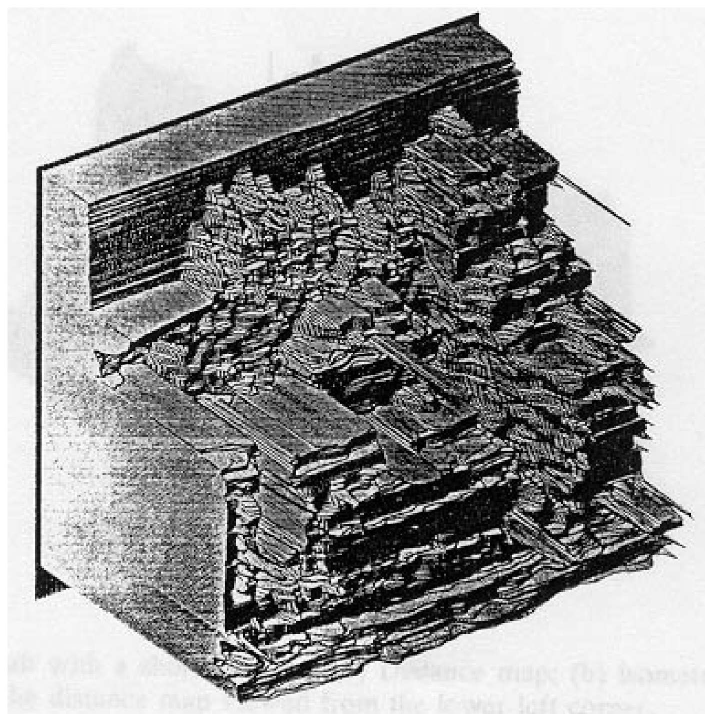
Example: reconstruct image from neighboring images



I1

I2

I10



Reprinted from “A Multiple-Baseline Stereo System,” by M. Okutami and T. Kanade, IEEE Trans. on Pattern Analysis and Machine Intelligence, 15(4):353-363 (1993). \copyright 1993 IEEE.

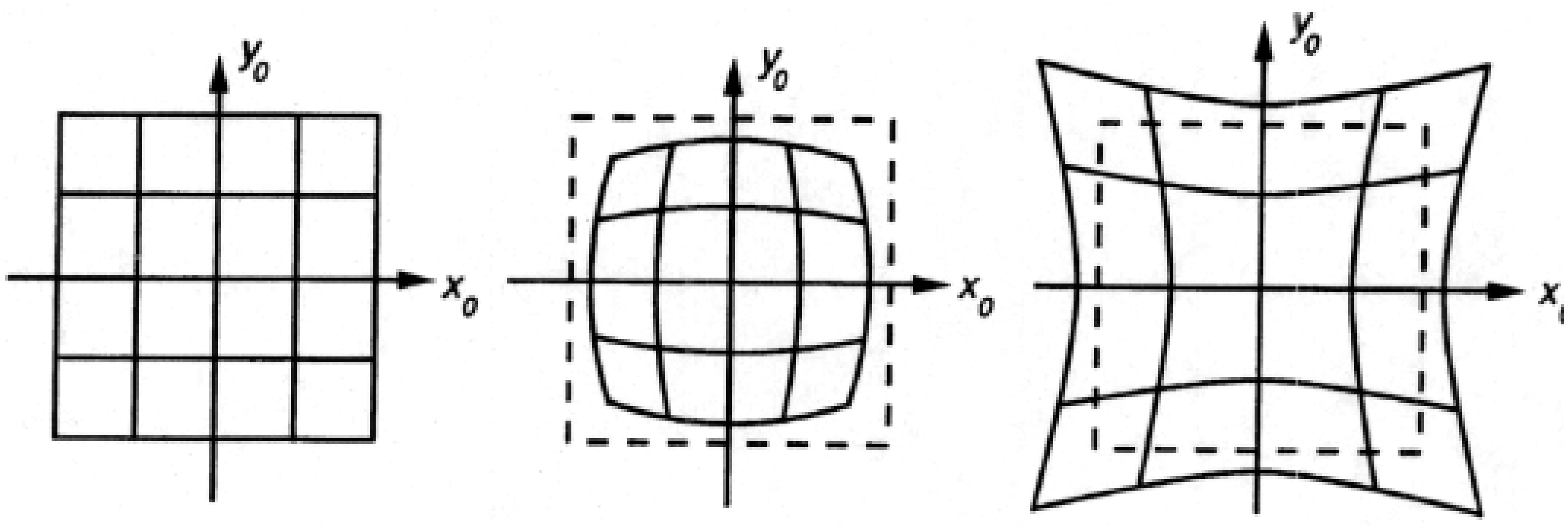
Geometric Distortions

- Our camera model is linear
- 3x4 linear matrix to relate 3D to 2D
  - Lines/planes map to lines/planes
- In real life there can be distortion
  - Due to the lens
- No longer linear
  - Lines/planes can become curves/surfaces

Radial distortion

- Points move based on their radius from the principal center
  - Where the principal axis meets the image plane
- Principal center may not be at the center of the image
- Distortion in cheap cameras can be worse

Radial Distortion



No Distortion

Barrel Distortion

Pincushion Distortion

Several models

- Most common is a polynomial in r
  - Assuming principal center to be at the center of the image
- $[x, y, w] = C[X, Y, Z, 1]$
- $[u,v] = [x/w, y/w]$
- $[u', v'] = [u, v] \pm d((u^2+v^2)^{1/2})$
- How to correct for it?
  - Complex optimization problems
  - Involves calibration patterns

END

Fundamental Matrix (**F**)

- A matrix **F** such that
  - $p'^T \mathbf{F} p = 0$
- Can be estimated from eight correspondences
- $L' = \mathbf{F}p$

  $L = \mathbf{F}^{\mathbf{T}}p'$
- $\mathbf{F}e_1 = 0$

  $\mathbf{F}^{\mathbf{T}}e_2 = 0$

Pure translation motion

- $\mathbf{F} = [e_2]_\mathbf{x}$
- If motion is parallel to x direction, then $e_2 = (1,0,0)$ (at infinity)
- $y = y'$
- Corresponding points are on the raster lines
  - Easy to find correspondences

Essential Matrix (**E**)

- Fundamental matrix of normalized cameras
  - K is identity
  - Use normalized coordinates and estimate **F**