Fourier Transform

Anecdote: 1807. Fourier claimed
"any cont. periodic signal can be
expressed as a sum of properly
chosen sinusoids."
Reviewed by Lagrange & Laplace.

opposed

vehemently

"cannot be used to represent
signals with corners."

For 15 years.

was published 15 yrs later only after
death of Lagrange.

Who was right?

Both were. You can get close to
signals with corner, but not exactly.
But the difference has zero
energy - proved by Gibbs & hence
Called Gibbs Effect.

However, this objection is true for
continuous domain only, for discrete
signals as in DSP, the representation
is exact & accurate.
Discrete Fourier Transform

Note: valid for periodic discrete signals that are infinite. So how do we apply to finite signals?
Assume the repeat — there are consequences of this assumption, which we will discuss later.

We will do the math & understanding in 1D, 2D is complex & you can depend on MATLAB. But all intuitions, insights, & understanding in 1D holds for higher dimensions.

Assume a signal with N samples:
\[ x[0...N-1] \]

DFT changes this to two arrays with \( N+1 \) samples —
\[ \Re x[0...N] \]
\[ \Im x[0...N] \]

\( x \) is said to be in time/spatial domain & \( \Re x \) & \( \Im x \) in freq. domain

\[ x[1...N] \]
\[ \text{DFT/Decomp} \]
\[ \text{Analysis} \]
\[ \text{Inv. DFT/Synthesis} \]
So, what do these $\text{Re} X$ & $\text{Im} X$ mean?

$\text{Re} X$ gives amplitude of $\frac{N+1}{2}$ cosine waves & $\text{Im} X$ gives amplitude of $\frac{N+1}{2}$ sine waves which when added together will create $X$.

"What is the freq. of the k\textsuperscript{th} cosine or sine wave?"

$\text{Re} X[k]$ signifies a sine wave that makes $k$ cycles over the $N$ samples.

P.S. Length of all these waves are $N$ also.

"$\text{Re} X[\frac{N-1}{2}]$ makes $\frac{N}{2}$ cycles over $N$ samples; $\approx 2$ samples per cycle."

"Maximum freq. Sine wave that can be represented by $N$ cycles."

Similarly for $\text{Im} X$.

Are there other way to express this?

(a) $k$ cycles.

(b) How many cycles per pixel? (f)

$$\frac{k}{N} = 0 - 0.5$$
(c) \[ f \times 2\pi = \omega \]

Called natural freq in radians.

What is the cosine wave whose amplitude is given by \( \text{Re} \times [k] \)?

Again, Cos wave is \( N \) samples long.

\[ c_k \hat{f} = \cos \left( \frac{2\pi k \hat{f}}{N} \right) \quad 0 \ldots N/2 \]

\[ = \cos \left( 2\pi \hat{f} \right) \quad 0 \ldots 0.5 \]

\[ = \cos \left( \omega \hat{f} \right) \quad 0 \ldots 1 \]

\[ x[i] = \sum_{k=0}^{N/2} \text{Re} \times [k] \cos \left( \frac{2\pi k i}{N} \right) \]

\[ + \sum_{k=0}^{N/2} \text{Im} \times [k] \sin \left( \frac{2\pi k i}{N} \right) \]

Linear combination of sines & cosines. Set \( \cos \) & \( \sin \) are called a basis —

i.e.(a) They are linearly independent of each other [one cannot be represented as a linear combination of another].

(b) Together they are necessary & sufficient to represent \( x \).

Note \( \text{Re} \times [k] \) is a cosine wave of
zero freq. \( G_0 = 1 \).

\[ \text{Re} \times \left[ e^{0j} \right] = \text{average of all samples} = \text{called DC component}. \]

\[ \sin 0 = 0 \]

\[ \therefore \text{Im} \times \left[ e^{0j} \right] = 0. \]

Also \( \sin \left( \frac{2\pi N i}{N} \right) \)

\[ = \sin (\pi i) = 0. \]

\[ \therefore N \text{ samples in time/space} \]

\[ \text{general} \quad N \text{ samples in freq.} \]

No in consistency.

So, how to compute \( \text{Re} \times \) \& \( \text{Im} \times \)?

How much of \( G_k \) \& \( S_k \) are contained in \( x \in \{0 \ldots N-1\} \)?

Correlation

\[ \text{Re} \times \left[ k \right] = \sum_{i=0}^{N-1} x[i] e^{0j} \cos \left( \frac{2\pi ki}{N} \right) \]

\[ \text{Im} \times \left[ k \right] = \sum_{i=0}^{N-1} x[i] e^{0j} \sin \left( \frac{2\pi ki}{N} \right) \]

Each \( k \) the basis sine \& cosine are linearly independent i.e. completely
Uncorrelated with the other.

Called orthogonal to each other.

Polar Notation

Note that for each \( f \), we get in DFT

\[
A \cos(f) + B \sin(f) = M \cos(f + \theta)
\]

where \( M = \sqrt{A^2 + B^2} \)

\[
\theta = \tan^{-1} \frac{B}{A}
\]

\( \frac{N+1}{2} \) cosine waves with amplitude & phase.

\[
\text{Mag} \times |k_j| = \sqrt{\text{Re}(k_j)^2 + \text{Im}(k_j)^2}
\]

\[
\text{Phase} \times |k_j| = \tan^{-1} \frac{\text{Im}(k_j)}{\text{Re}(k_j)}
\]

Why cosine & not sine?

Sine waves cannot represent DC.

Now much easier to visualize?

Amplitude gives content info, &
Synchronized phase is an edge
In 2D plots, 2D (f,0).

\[ x_{EiJ} = \sum_{k=0}^{N/2} \left( \text{Re} x[k] \cos \left( \frac{2\pi \cdot k}{N} \right) ight. \\
\left. + \text{Im} x[k] \sin \left( \frac{2\pi \cdot k}{N} \right) \right) \]

But slightly different

\[ \text{Re} x[k] = \text{Re} x[k] \]
\[ \text{Im} x[k] = \text{Im} x[k] \]

except for

\[ \text{Re} x[0] = \text{Re} x[0] \]
\[ \text{Re} x[N/2] = \text{Im} x[N/2] \]

Think of these as scale factors or normalization factors related.
to the difference between analog and digital.

Note that each wave represents a discrete freq of \( k = 0, \frac{N}{2} \) indicating a wave that makes \( k \) cycles per \( N \) samples.

In freq domain plot

Note each wave of \( k \) cycles per \( N \) samples represent a bandwidth in the spectral plot.

Note for \( k = 1 \ldots \left( \frac{N}{2} - 1 \right) \) the waves represent a bandwidth \( + \frac{2}{N} \). For \( k = 0 \) & \( \frac{N}{2} \) it is \( \frac{1}{N} \).

Think of these as multiplicative scale factors to account for the
difference in spectral density.

Polar Nuisances

(a) Remember phase is in radians.
(b) If phase is 90° or -90°,

\[ \Re x[k] = 0 \]

:. Divide by 0 when converting to polar notation.

(c) If \( \Re x[k] = 1 \), \( \Im x[k] = 1 \)

\[ \text{Phase} = \tan^{-1} 1 = 45° \]

If \( \Re x[k] = -1 \), \( \Im x[k] = 1 \)

\[ \text{Phase} = \tan^{-1} 1 = 135° \]

:. \( \pi \) ambiguity.

Whenever phase is below \( -\pi \), it snaps to \( \pi \). Since \( \theta, \theta + 2\pi \), \( \theta + 4\pi \) are same.

Called Cenwrapping of Phase.

(d) \( -\pi \) & \( \pi \) are same phase shift.

Can cause discontinuities.
The biggest property of PT is is duality (note the strikingly similar synthesis & analysis equation). This gives rise to several interesting properties.

Properties:

1) Homogeneity

\[ x[i] \Rightarrow x[f] \]
\[ kx[i] \Rightarrow kx[f] \]

2) Additivity

\[ x_1[i] \Rightarrow Re x_1[f], Im x_1[f] \]
\[ x_2[i] \Rightarrow Re x_2[f], Im x_2[f] \]
\[ x_1[i] + x_2[i] \Rightarrow Re x_1[f] + Re x_2[f], Im x_1[f] + Im x_2[f] \]

P.S. Addition cannot be done in polar coordinates. Why?

3) Phase

\[ x[i] \Rightarrow Mag x[f], Ph x[f] \]
\[ x[i+90] \Rightarrow Mag x[f], Ph x[f] + 90^\circ \]

Signals symmetric about center have zero phase.
\[ A - B = - \text{Phase} \]

- They cancel to give zero phase.

DFT views as circular. Also any signal that is symmetric about any pt. (may not be center) has linear phase.
\[ x[i] \Rightarrow \text{mag} \times [f] \quad \phi \times [f] \]
\[ x[-i] \Rightarrow \frac{\text{mag} \times [f]}{\phi \times [f]} \]
\[ = x \ast [f] \]

Called Complex Conjugate.

\[ x[i] \Rightarrow x[f] \]
\[ x[*] \Rightarrow y[f] \]

What does multiply mean?

\[ \text{mag} \times [f] \times \text{mag} \times y[f] \]
\[ \phi \times [f] + \phi y \times [f] \]

What is nonlinear phase?

Non-linear features superimposed on linear phase.

Exploring The duality

a) Compress in time \(\Leftrightarrow\) expand in frequency

\& vice versa

\[ \xrightarrow{\text{FT}} \]

\[ t \rightarrow f \]
This is the basis of LPF.

2x2 kernel * Image

4x4 kernel * Image

Widening in time domain leads to more severe frequency rejection.

b) Multiplication (≡) Convolution

Vice versa

Eq. Amplitude Modulation

What is happening in frequency domain?
At the receiver if we multiply the signal with the appropriate filter with center at $f$, we can reconstruct the signal.

This is the basic phenomenon behind radio. Different stations use different carrier frequencies but can transmit in parallel since each will fall in a different ultrasonic range.

**FM**

Frequency modulation, modulates the frequency of the transmitted signal based on the amplitude of the signal.

Say choose 65 kHz depending on amplitude of signal.
Say will go between 55 - 75 kHz.

- In frequency domain, it will occupy a band of 55 - 75 kHz.

Different stations can use different carrier frequency again to occupy different bands.

**Fourier Pairs**

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Freq. Domain</th>
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</thead>
<tbody>
<tr>
<td>a) Impulse</td>
<td>a) Constant</td>
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<tr>
<td>b) Box filter</td>
<td>b) $\sin c$</td>
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<tr>
<td></td>
<td>$AT \neq T = \frac{\sin(\frac{\pi}{T})}{\frac{\pi}{T}}$</td>
</tr>
<tr>
<td></td>
<td>Infinite</td>
</tr>
<tr>
<td></td>
<td>Cannot make perfect UF</td>
</tr>
</tbody>
</table>
Time Domain                      Free Domain

\( \text{Sinc} \)                      \( \text{Box} \)

But how can you use an infinite kernel.

\( \text{Gaussian} \)                  \( \text{Gaussian} \)

\( \text{Gaussian} \times \text{Synch} \)        \( \text{Gaussian} \times \text{Box} \)
\( \text{(Make it finite)} \)                      \( \text{Gives a better cut off} \).