Note Title 9/15/2008

Fourier Transform

Anecdote: 1807 Fourier claimed "any cont. periodic signal can be expressed as a sum of properly chosen sinusoids".

Reviewed by Lagrange & Laplace.

Lagrange: opposed vehemently "cannot be used to represent signals with corners" For 15 years.

Laplace: publish

Was published 15 yrs later only after death of Lagrange.

Who was right?

Both were. You can get close to signals with corner, but not exactly. But the difference has zero energy - proved by Gibbs & hence Called Gibbs Effect.

However, this objection is true for continuous domain only, for discrete signals as in DSP, the representation is exact & accurate.

Discrete Fourier Transform

Note valid for periodic discrete signals that are infinite. So how do we apply to finite signals? Assume the repeat — There are consequences of this assumption, which we will discuss later.

We will do the math & understanding in 1D. 2D is complex & you can depend on MATLAB. But all intuitions, insights & understanding in 1D holds for higher dimension.

Assume a signal with N samples.
$x[0 \ldots N-1]$

DFT changes this to two arrays with $\frac{N}{2}+1$ samples — $Re\, x[0 \ldots \frac{N}{2}]$

$Im\, x[0 \ldots \frac{N}{2}]$

x is said to be in time/spatial domain & $Re\, x$ & $Im\, x$ in freq. domain

Time/Space $\qquad\qquad\qquad\qquad$ Freq

$x[1 \ldots N]$ $\xrightarrow{\text{DFT/Decomp/Analysis}}$ $Re\, x[0 \ldots \frac{N}{2}]$

$Im\, x[0 \ldots \frac{N}{2}]$

Inv. DFT/Synthesis

So, what do these Re X & Im X mean?

Re X gives amplitude of $\frac{N}{2}+1$ cosine waves & Im X gives amp of $\frac{N}{2}+1$ sine waves which when added together will create x.

$\therefore$ What is the freq. of the k^{th} cosine or sine wave?

Re X [k] signifies a sine wave that makes k cycles over the N samples.

P.S. length of all these waves are N also.

$\therefore$ Re X $[\frac{N}{2}]$ makes $\frac{N}{2}$ cycles over N samples $\therefore$ 2 samples per cycle.

$\therefore$ Max$^{\underline{m}}$ freq. sine wave that can be represented by N cycles.

Similarly for Im X.

Are there other way to express this?

(a) k cycles.

(b) How many cycles per pixel? (f)

$\frac{k}{N}$ $\therefore$ 0 – 0.5

(c) $f \times 2\pi = \omega$

Called Natural freq in radians.

∴ What is the cosine wave whose amplitude is given by Re $X[k]$?

Again, Cos wave is N samples long.

$$\therefore \; c_k[i] = \cos\left(\frac{2\pi k i}{N}\right) \quad 0 \ldots \frac{N}{2}$$

$$= \cos(2\pi f i) \quad 0 \ldots 0.5$$

$$= \cos(\omega i) \quad 0 \ldots \pi$$

$$x[i] = \sum_{k=0}^{\frac{N}{2}} \mathrm{Re}\,X[k] \cos\left(\frac{2\pi k i}{N}\right)$$

$$+ \sum_{k=0}^{N/2} \mathrm{Im}\,X[k] \sin\left(\frac{2\pi k i}{N}\right)$$

∴ Linear combination of Sines & Cosines.

Set of Cos & Sin are called a basis —

i.e (a) They are linearly independent of each other [one cannot be represented as a linear combination of another].

(b) Together they are necessary & sufficient to represent x.

Note Re $X[0]$ is a cosine wave of

Zero freq. $\cos 0 = 1$.

$\therefore$ Re $X[0]$ = average of all samples – called DC component.

$\sin 0 = 0$

$\therefore$ Im $X[0] = 0$.

Also $\sin\left(\frac{2\pi \frac{N}{2} i}{N}\right)$

$= \sin(\pi i) = 0$.

$\therefore$ N samples in time/space generate N samples in freq.
No inconsistency.

So, how to compute Re X & Im X?

How much of C_k & S_k are contained in $x[0..N]$?

Correlation

$$\text{Re}\,X[k] = \sum_{i=0}^{N-1} x[i] \cos\left(\frac{2\pi k i}{N}\right)$$

$$\text{Im}\,X[k] = \sum_{i=0}^{N-1} x[i] \sin\left(\frac{2\pi k i}{N}\right)$$

Each of the basis sine & cosine are linearly independent i.e. completely

uncorrelated with the other.
$\therefore$ Called Orthogonal to each other. $\therefore$ Forms a Basis.

Polar Notation

Note that for each f, we get in DFT

$A \cos(f) + B \sin(f)$
$= M \cos(f + \theta)$
where $M = \sqrt{A^2 + B^2}$
$\theta = \tan^{-1} \frac{B}{A}$

$\therefore \frac{N}{2}+1$ Cosine waves with amplitude & phase.

$\text{Mag } X[k] = \sqrt{\text{Re } X[k]^2 + \text{Im } X[k]^2}$
$\text{Phase } X[k] = \tan^{-1} \frac{\text{Im } X[k]}{\text{Re } X[k]}$

Why Cosine & not Sine?
Sine waves cannot represent DC.

Now much easier to visualize?
Amplitude gives content info, &

phase provides location info.
Synchronized phase is an edge
In 2D plots, 2D (f, θ).

SLIDES HERE

Synthesis

We wrote earlier

$$x[i] = \sum_{k=0}^{N/2} \left(Re\,X[k] \cos\left(\frac{2\pi i k}{N}\right) + Im\,X[k] \sin\left(\frac{2\pi k i}{N}\right) \right)$$

But slightly different

$$Re\,\bar{X}[k] = \frac{Re\,X[k]}{N/2}$$

$$Im\,\bar{X}[k] = \frac{Im\,X[k]}{N/2}$$

for all k,

except for

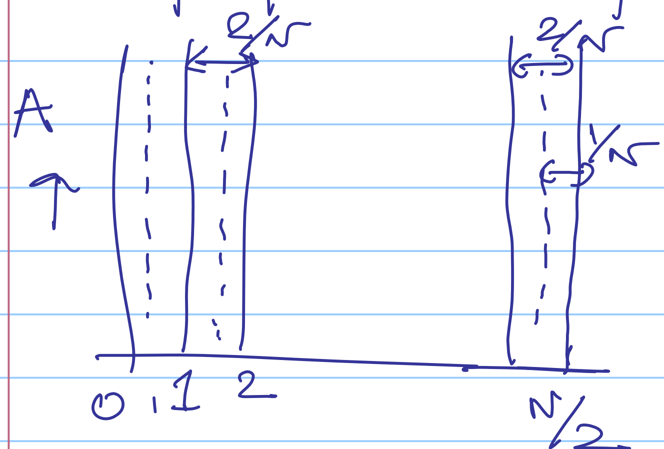
$$Re\,\bar{X}[0] = \frac{Re\,x[0]}{N}$$

$$Re\,\bar{X}\left[\frac{N}{2}\right] = \frac{Im\,x[N/2]}{N}$$

Think of these as scale factors or normalization factors related to The difference between analog & digital.

Note that each wave represents a discrete freq of $k=0 - \frac{N}{2}$ indicating a wave that makes k cycles per N samples.

In freq domain plot



Dotted lines shows the bands of which each wave is the center.

Note each wave of k cycles/N samples represent a bandwidth in the spectral plot.

Note for $k = 1 \cdots \left(\frac{N}{2} - 1\right)$ the waves represent a bandwith of $2/N$. For $k=0$ & $N/2$ it is $1/N$.

Think of these as multiplicative scale factors to account for the difference in spectral density.

Polar Nuisances

a) Remember phase is in radians.

b) If phase is $90°$ or $-90°$,

$Re\,X[k] = 0$

$\therefore$ Divide by 0 when converting to polar notation.

c) If $Re\,X[k] = 1$, $Im\,X[k] = 1$

$\therefore$ Phase $= \tan^{-1} 1 = 45°$

If $Re\,X[k] = -1$, $Im\,X[k] -1$

Phase $= \tan^{-1} 1 = 135°$

$\therefore \pi$ ambiguity.

Whenever phase is below $-\pi$, it snaps to π. Since θ, $\theta + 2\pi$, $\theta + 4\pi$ are same.

Called unwrapping of phase.

d) $-\pi$ & π are same phase shifts.

Can cause discontinuities.

The biggest property of FT is is duality (note the strikingly similar synthesis & analysis equation.). This gives rise to several interesting properties.

Properties

a) Homogeneity

$x[i] \Rightarrow X[f]$

$kx[i] \Rightarrow kX[f]$

b) Additivity

$x_1[i] \Rightarrow \mathrm{Re}\,X_1[f] \quad \mathrm{Im}\,X_1[f]$

$x_2[i] \Rightarrow \mathrm{Re}\,X_2[f] \quad \mathrm{Im}\,X_2[f]$

$\therefore x_1[i] + x_2[i] \Rightarrow \mathrm{Re}\,X_1[f] + \mathrm{Re}\,X_2[f],$
$\mathrm{Im}\,X_1[f] + \mathrm{Im}\,X_2[f]$

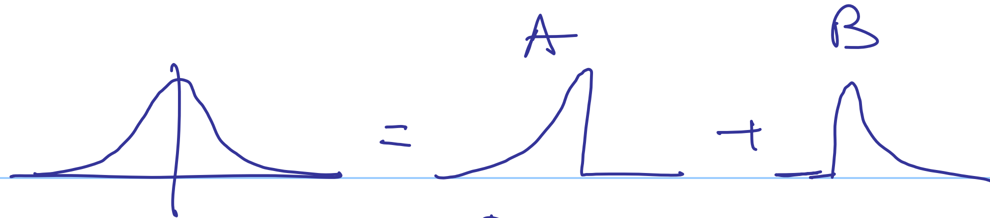
P.S. Addition cannot be done in polar coordinates. Why?

c) Phase

$x[i] \Rightarrow \mathrm{Mag}\,X[f] \quad \mathrm{Ph}\,X[f]$

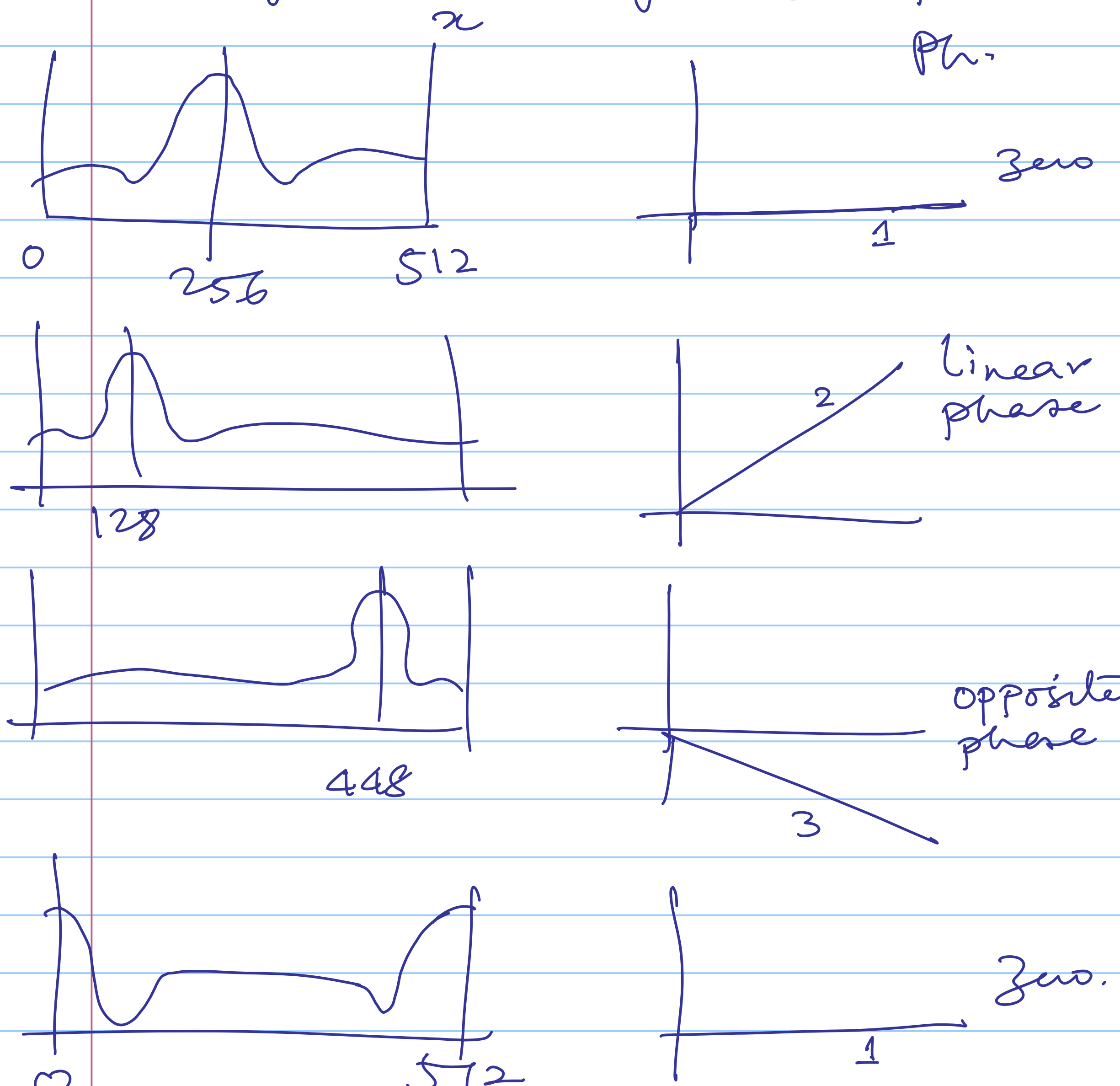
$x[i+s] \Rightarrow \mathrm{Mag}\,X[f] \quad \mathrm{Ph}\,X[f] + 2\pi f s$

Signals symmetric about center has zero phase.



Phase of A = - Phase of B.

∴ They cancel to give zero phase.



DFT views as circular. Also any signal that is symmetric about any pt. (may not be center) has linear phase.

$x[i] \Rightarrow$ Mag $X[f]$ & Ph $X[f]$
$= X[f]$

$x[-n] \Rightarrow$ Mag $X[f]$ & $-$Ph $X[f]$
$= X^*[f]$

Called Complex Conjugate.

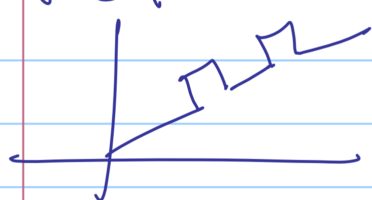
$$x[i] \Rightarrow X[f]$$
$$*\qquad\qquad\qquad\qquad \times$$
$$y[i] \Rightarrow Y[f]$$

What does multiply mean?

Mag $X[f] \times$ Mag $Y[f]$

Ph $X[f]$ + Ph $Y[f]$

What is nonlinear phase?

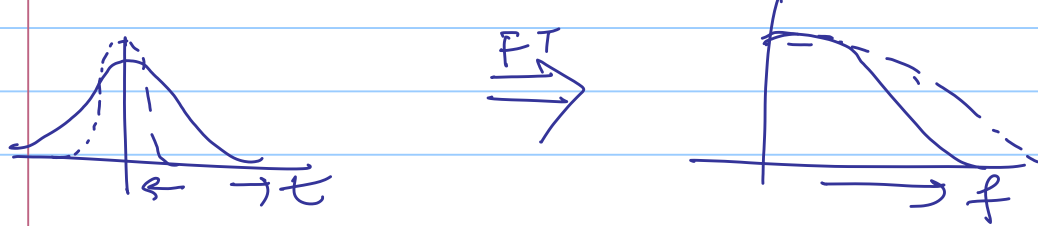


Non-linear features superimposed on linear phase.

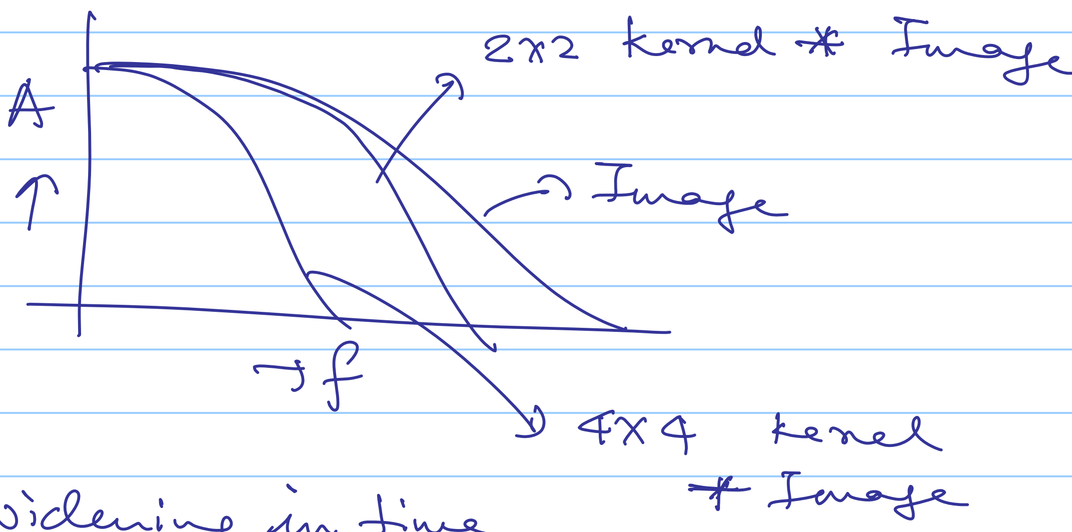
Exploring the duality

a) Compression in time $\Longleftrightarrow$ Expansion in Frequency

& vice versa



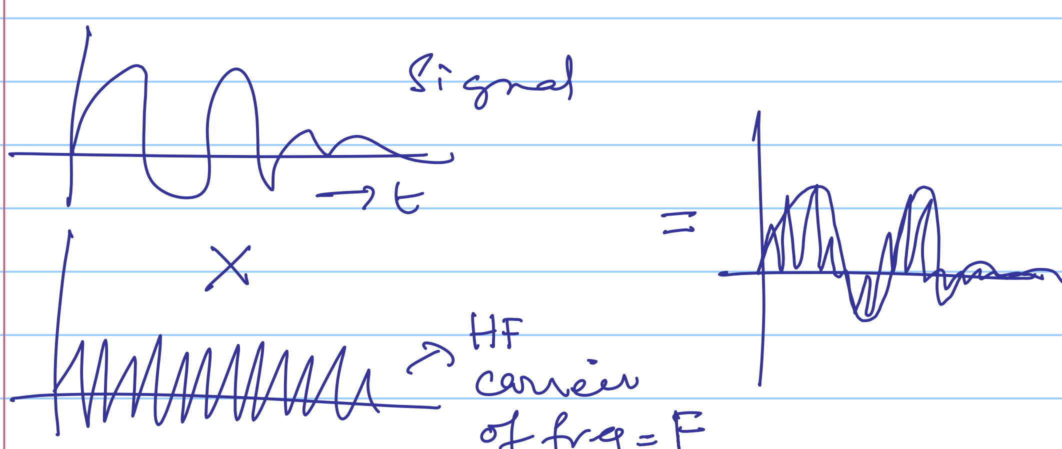
This is the basis of LPF.



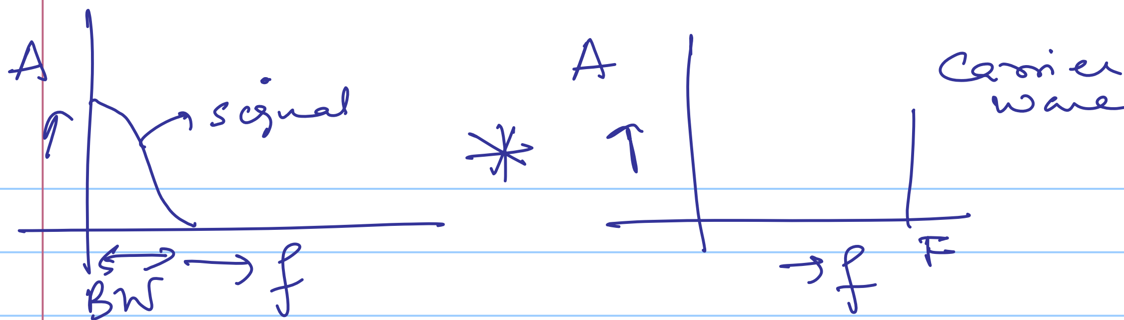
Widening in time domain leads to more severe frequency rejection.

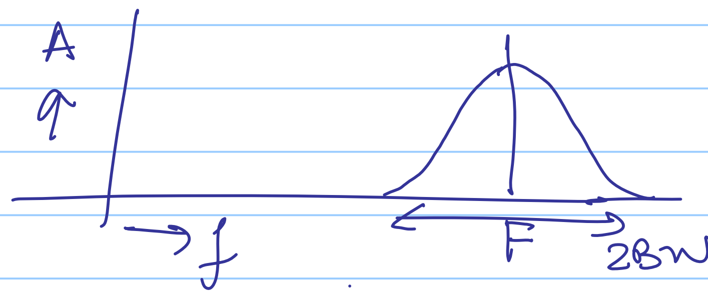
b) Multiplication ⟺ Convolution
vice versa

Eg. Amplitude Modulation



What is happening in frequency domain?





At The reciever if we multiple FT with the apporpiate filter with center at F, we can reconstruct the signal.

This is the basic phenomenon behind radio. Different stations use diffrent carrier frequencies but can transmit in parallel since each will fall in a diffent ultrasonic range.

FM

Frequency modulation, modulates the frequency of The transmitted Signal based on the amplitude of the signal.

∴ Say choose 65 KHz
Depending on amplitude of signal

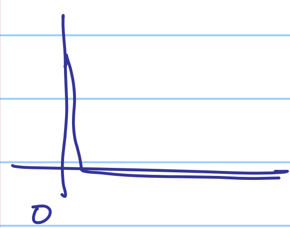
say will go between 55 – 75 kHz.
∴ In frequency domain, it will occupy a band of 55 – 75 kHz

Different stations can use different carrier frequency again to occupy different bands.

Fourier Pairs

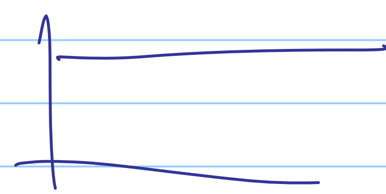
Time Domain

a) Impulse



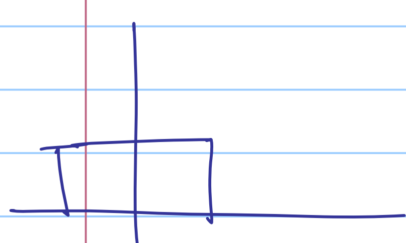
Freq. Domain

a) Constant



Time Domain

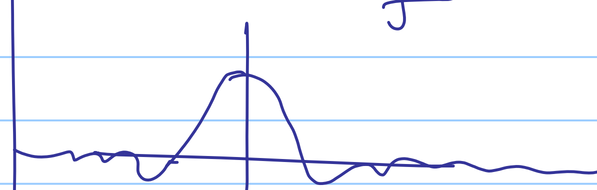
b) Box filter



Freq. Domain

b) Sinc

$$A[f] = \frac{\sin(f)}{f}$$



Infinite.

∴ Cannot make a perfect LPF

Time Domain	Freq Domain
c) Sinc	Box Perfect LPF.
But how can you use an infinite kernel.	
d) Gaussian	Gaussian
∴ Gaussian $\times$ Synch (Make it finite)	Gaussian $*$ Box Gives a better cut off.