

Sketching Free-form Surfaces Using Network of Curves

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Abstract

This work addresses the issue of generating free-form surfaces using a 2D sketch interface. As the first step in this process, we develop a methodology to sketch 3D space curves from 2D sketches. Since the inverse projection from 2D sketches to 3D curve or surface is a one to many function, there is no unique solution. Hence we propose to interpret the given 2D curve to be the projection of the 3D curve that has minimum curvature among all the candidates in 3D. We present an algorithm to efficiently find a close approximation of this minimum curvature 3D space curve. In the second step, this network of curves along with the boundary information are given to the surface fitting method to generate a free-form surfaces.

1. Introduction

Modeling by sketching has a fundamental problem of interpreting the 2D sketches of curves and surfaces as a 3D entity. Most of the real world objects have non-planar 3D curves and these curves are the fundamental component of surface generation [IMT99, ZHH96] for 3D models. Space curves are also used as input for specifying implicit surfaces and also for manipulating and modifying object deformations [SF98, IH03]. The problem of finding the 3D space curve from a 2D sketch is mathematically indeterminable and has many possible solutions; it is well known that the fundamental difficulty is in choosing the appropriate one. Most of the sketch-based modeling tools try to avoid addressing this fundamental problem directly due to its complexity. These systems provide a fixed canvas to the user to draw the 2D curve, thus fixing the 3D curve to be on the canvas. They provide tools to change the shape of the canvas to enable the user to draw non-planar curves [IITS04]. One of the clever uses of this canvas method was by Igarashi et al. [IMT99] where the system provides a planar canvas first and the user starts with a planar closed curve. A round object is generated using that planar closed curve and from then on, the generated object becomes the canvas for all future curve drawings. Often, curves that are not drawn on the canvas are interpreted as gestural inputs for viewpoint or object manipulation. In spite of all the successes that have been achieved in the field of sketch based modeling, the fundamental problem of space curve determination still remains to be a challenging one.

In our paper, we attempt to provide an intuitive and mathematically sound insight into this problem of 3D space curve

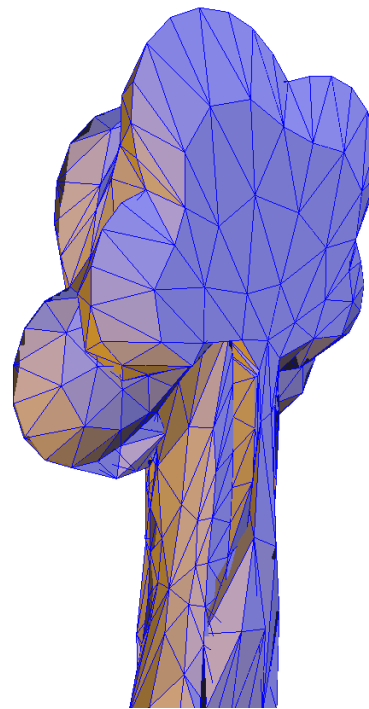


Figure 1: A sample model generated by the system.

determination from a 2D curve. We propose that the 3D curve corresponding to the given 2D curve to be the one that minimizes the maximum normal curvature in 3D. Cur-

vature minimizing interpolation approaches have been previously used in the study of visual perception to create seamless color smoothing for overlapping multi-projector displays [MS05], indicating a relationship between curvature minimization and perception of desirability and intuitiveness. In this paper we take a similar approach to address the following problem: given a 2D curve, the projection parameters, and the depth range, find the 3D space curve within the given depth range that has the minimum curvature and matches with the 2D curve under the given projection.

The second issue we address in this paper using is the generation of surfaces using the above curvature minimizing space curves. We sketch the network of planar and non-planar input curves and specify the patch connectivity. Our system automatically generates a free form surface that fits these network of curves. Once an initial 3D model, based on the input sketch is presented, the user could modify it to get the desired shape. In our current exposition, we stress more on the proof-of-concept implementation of geometric principles and less on the user-interface design.

Specifically, following are the main contributions of this paper:

1. We prove the existence of the curvature minimizing space curve for an orthographic 2D projection of a polynomial curve.
2. Given only a sequence of 2D curve points as input, we present an efficient algorithm to best approximate the curvature minimizing 3D space curve, from that viewpoint. We use Laplacian filter effectively along with our approach to achieve the desired result of curvature minimizing depth interpolation.
3. We generate smooth free form surfaces using minimal user input using an iterative method having fast convergence. We present a proof-of-concept system that fits a smooth surface given the boundary curves that are sketched using our system.

2. Related Work

There has been a lot of ongoing research on generating 3D models using a sketching interface. The different endeavors in the field of 3D curves focus either on *curve manipulation methods* or on *2D to 3D curve generation methods*. The approach mentioned in this paper belongs to the latter category.

Existing space curves can be manipulated indirectly in many commercial packages like Maya, 3DStudioMax and most other CAD tools, by modifying the curve parameters such as spline control points or knot values. Spline curves can also be manipulated directly as described in [FB93, GA98]. Modifying existing curves by over-sketching was initially presented by Baudel [Bau94]. This approach has been adopted for curve modification and manipulation by several other researchers in this field [CMZ*99, FRSS04]. Cohen et al. [CMZ*99] used the

correlation between the curve and its shadow as a visual cue to help the user compute and modify the shape of the 3D curve. Karpenko et al. [KHR04] take a multi-view sketching approach using epipolar lines as cues to deduce shape information. Their method uses perspective projection and can determine the 3D values only when drawn from two different view points. The epipolar geometry approach works for simple cases but fails to give desirable results for more complex objects due to the nature of their algorithm.

In the recent past, there has been some research devoted to extract depth information from 2D curves and strokes directly. Most of the existing literature on space curve generation relies on the domain specific knowledge. The edge-vertex graph method has been utilized to extract the depth information in systems best suited for architectural designs and CAD type applications in [LS96, SC04]. Furthermore, [LS02] use an interesting domain specific learning based approach using geometric correlation to deduce the 3D model that is again more suitable for CAD tools. The underlying assumption behind this learning technique is that the input sketch comprises of straight lines and thus does not account for free-form curves. It also follows from the paper that the human ability to perceive the 3D information from 2D sketch is based on prior knowledge about the drawn object or scene and human perception fails to reconstruct random objects. Learning based methods are also non-interactive and are computationally expensive.

One of the earliest example of constructing free-form space curves is the 3-Draw system [SRS91] that extracts the 3D coordinate information by using a tracker-based system. A curvature minimizing approach was followed by [PK89]. Ijiri et al. [IITS04] utilize the input sketch to generate a curved canvas where the drawn curve essentially becomes an extruded surface from a given viewpoint providing the user the capability of drawing non-planar curves on that surface. Tolba et al. [TDM99] use the approach of oriented projective representations of 2D points. This allows "3D-like" perspective viewing, object manipulation and rendering of the 2D points, but they do not generate actual 3D values from 2D points.

To the best of our knowledge, most of the existing approaches about space curve generation are based on heuristics and there is no literature on research that performs domain independent mathematical treatise to generate intuitive free form space curve. Our paper attempts to address this issue by using curvature minimizing depth interpolation. We present an interpolation method to generate artistic and expressive 3D curve from a *single* 2D curve input. It is obvious that one cannot provide the exact curve that user has in mind, just from a single 2D sketch. Hence, any curve *manipulation* technique can be used thereafter to modify our computed 3D curve. The method discussed empowers the user by simplifying the user-interface, quickens the system learning time of the user and overall time to sketch the object.

Recently, some papers have been presented on fitting surfaces to a network of curves. Teddy [IMT99] is one of the exemplary works in the field of free-form surface generation using polygonal representation. Michalik et al. [MKB02], on the other hand used a parametric representation of surface using a constraint based design. Volumetric representation is also used effectively by researchers [ONNI03] which has the advantage of relaxing the topological limitations on surface generation. Furthermore, recent research on surface generation techniques use implicit surface representation. Notably, Tai et al. [TZF05] obtain an analytical convolution surface from sketched silhouettes. In [SWZ04], Schaefer et al. make clever use of lofting techniques to create subdivision surfaces from curve network.

Our system generates free form surfaces from sketched network of curves by using a simple topological triangulation followed by surface fairing.

3. Space Curve Generation

We repeat the problem statement for the sake of completion. *Given a 2D curve, the orthographic projection parameters, and the depth range, the problem is to find from all the possible 3D space curves within the given depth range and that matches with the 2D curve under the given projection, the curve that minimizes the maximum curvature.* The given 2D curve is assumed to be a polynomial curve. Since any polynomial basis can be converted to and from a Bernstein basis, we can convert the given polynomial curve into a 2D Bezier curve. We state, prove and use a theorem given in the appendix involving the control points of Bezier curves to come up with a solution for the stated problem.

Theorem 1 Given a 2D Bezier curve with a known depth range, the depth of the control points of curvature minimizing 3D Bezier curve is monotonically and equally spaced within the given depth range.

If we use this theorem to construct the 3D curve from the 2D sketch, it will have seemingly two drawbacks: it can produce only monotonic curves and low curvature curves. In the problem of going from 2D to 3D, it is obvious that we have to constrain the problem to fix one solution from infinite number of solutions. Monotonicity from a given view point is a natural constraint that is required by a mathematically convincing choice of the solution – minimum curvature curve. At the same time, the generated curve can be non-monotonic in every other viewing direction and hence is not a limitation. Second, the curvature is minimized within certain constraints - the given depth range and the 2D projection. Smaller depth ranges provide less space to reduce the curvature. Hence arbitrarily high curvature curves can be created by using appropriately small depth ranges (Figure 3(b)).

3.1. Our Algorithm

The 2D curves drawn in sketching applications are not usually Bezier curves, but a sequence of edge connected points. Hence, we have to adapt our stated theorem for the depth extraction for a generic 2D curve. One way to adapt our result on curvature minimizing depth interpolation is to fit a Bezier curve to the input point sequence and interpolate the depth of this 2D Bezier curve. This is a computationally expensive operation and most importantly, the error in curve fitting will provide a distracting and unexpected feedback to the user.

In this section we provide an alternative method for finding the depth of the sketched 2D curve points that approximates the minimum curvature 3D curve. We use the following fundamental observation of the 3D Bezier curve chosen in the previous section (but not true in a generic Bezier curve): since the depth values of the control points are equally spaced, the closest point on the curve from any control point has the same depth value as the control point. Hence if we identify these *key points* on the curve that are closest to the control points and space their depth values equally, we will have the first approximation of the space curve. Since we do not have these 3D Bezier curve control points and hence of its 2D projection, identifying even the correct *number of key points* is difficult, let alone the actual key points. Even though we do not have an explicit Bezier representation of the 2D sketched curve, the number of critical points in the curve is the lower bound on the number of control points of the hypothetical Bezier representation of the same curve. The critical point need not be the closest curve point (key point) to the corresponding control point. The fundamental source of *approximation* in our algorithm for curvature minimizing depth interpolation comes from the fact that we assume that these critical points are key points.

The second source of approximation comes from the fact that, there need not be a critical point corresponding to every control point. Hence all the required control points cannot be found by just one sequence of critical points. We solve this issue by subdividing the original 2D curve at the initial critical point (*key point*) locations. If we perform a de-Casteljau subdivision of the Bezier curve at the *key points*, the above theorem still holds good for each of the subdivided curve segments. In this process, we increase the total number of control points (sum of the control points of the subdivided curve segments) and hence the possibility of more critical points. Using this observation, we repeat the process of finding the critical points recursively within each 2D segment between the initial sequence of critical points up till a threshold in terms of linearity of the curve segment. Since we assume that all these critical points are representatives of *key points*, we assign equally spaced depth values within the range to the critical points. The depth of all other points on the 2D curve is computed by linearly interpolating the depth between the critical points based on the curve length parameterization.

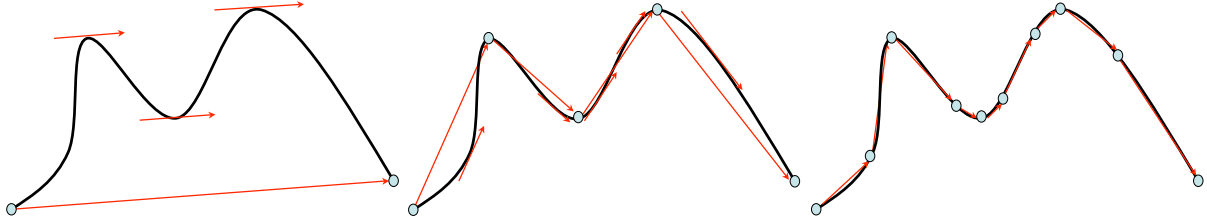


Figure 2: Illustration of our approximation algorithm to generate curvature minimizing space curve. At every iteration the critical points of the distance function from the line joining the first and last points of the curve segment is chosen for further curve subdivision. The recursion ends when the curve segment can be approximated well by the line joining its end points.

The resulting 3D space curve, though a very close approximation of the curvature minimizing curve, has non-smooth, high frequency transitions of the depth values across the critical points. We use the Laplacian filter to obtain a final smooth space curve as it is extremely fast and simple as required by interactive sketching applications. But it is a known fact that this filter produces shrinkage effect for geometric primitives. In our algorithm, the shrinking of curves is prevented by keeping the depth at the curve end points constant. We apply a few iterations of this smoothing filter *only* to the interpolated depth values of all the interior points on the curve, as $z_i' = z_i + \lambda(\Delta z_i)$ where $\Delta z_i = ((z_{i-1} + z_{i+1})/2) - z_i$ and $\lambda < 1.0$ for a low-pass filter. This produces an excellent approximation of the curvature minimizing curve Figure 3.

The summary of the algorithm for finding the curvature minimizing curve is given as pseudo-codes in Algorithms 1 and 2. An illustration of this algorithm is shown in Figure 2.

Algorithm 1 ReturnCriticalPoints(Curve C)

```

{Function: Find the Critical Points of the 2D curve C.}
{Given: Sample points of a 2D curve C}
L = Line between the first and the last sample points of C.
CS = Sequence of all the critical points of the distance
function of C from L.
R = Subset of CS, that is within a user-defined threshold
distance from L.
CS = CS - R.
if CS ==  $\phi$  then
    return; {Terminating condition.}
end if
CriticalList = CS
CheckList = {First point of C}  $\cup$  CS  $\cup$  {Last point of C}
for every curve segment  $C_i$  between consecutive points in
CheckList do
    CriticalList = CriticalList  $\cup$  ReturnCriticalPoints( $C_i$ ) {
    Insert into CriticalList, the critical points of  $C_i$  in order
of their appearance in C.}
end for
return CriticalList;

```

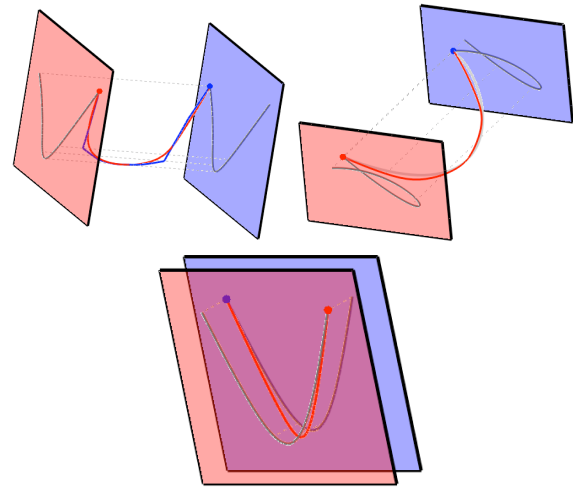


Figure 3: (a) Effect of filtering. Blue curve shows the interpolation before filtering and the red curve shows the curve after filtering. (b) shows reconstruction of space curves using our approximate method for different Bezier curves. The actual curvature minimizing Bezier curve as suggested in Theorem 1 is shown in gray. Our approximate curve follows the actual curve closely. Notice in (b) that even from a viewpoint where projections are self intersecting in 2D, the reconstructed curvature minimizing curve follows the actual Bezier curve. Our algorithm can also generate high curvature curve(c)

4. Surface Generation

Our system generates free form surfaces from a network of input curves. In our current implementation, once the user sketches the desired shape of the object in the form of outline curves, he/she has to specify the connectivity of the face forming the surface. We intend to automate the process of patch identification. Once the patch boundary connectivity is known, our system then generates a smooth free-form surface based on the given boundary curve network. In order to maximize the number of regular vertices (here degree 5) we

Algorithm 2 CurvatureMinimizingCurve(Curve C, Depth z0, Depth z1)

```
{Function: Curvature Minimizing Interpolation of Depth}
{Given: Sample points of a 2D curve C; depth values of
the first and the last points of C, z0 and z1. }
CriticalList = {First point of C} ∪ ReturnCritical-
Points(C) ∪ {Last point of C}
Let DepthInterpC be the points in C with unset depth val-
ues.
Interpolate the depth value linearly between z0 and z1
among all points of CriticalList and set their values in
DepthInterpC.
for every curve segment  $C_i$  between consecutive points in
CriticalList do
    Interpolate the depth value based on curve length pa-
    rameterization and set their values in DepthInterpC.
end for
CurvMinC = LaplacianFilter(DepthInterpC)
return CurvMinC
```

first find the additional vertices in the interior of the patch and their connectivity and then find the geometric coordinates for these vertices. Our surface generation method consists of three steps: normal computation and interpolation, topological triangulation in which we find the connectivity, and finally, surface fairing in which we find the geometry. We briefly describe each of these steps below.

Normal Computation and Interpolation: Every corner vertex in the input closed boundary curve sequence has at least degree three. Hence a normal vector with a stable local coordinate system can be computed at these corner vertices. Then these coordinate systems are interpolated using quaternions at other intermediate vertices on the boundary curve. We use the publicly available spherical linear interpolation, *slerp* [Sho] for this purpose. In order to reduce the noise in normal estimation, the input segments which suffer from sharp transitions were smoothed.

Topological Triangulation: The input to this step is a closed sequence of connected curves. Curves are represented as piecewise linear segments. The output is a topology of the triangulation (with unrefined geometric coordinates for the vertices) that consists of input vertices on the curve and the newly introduced vertices in the interior of the patch. The algorithm considers the sequence of curves as one single piecewise linear closed (boundary) curve. The goal is to create a new piecewise linear closed curve inside the patch, such that the new curve has half the number of vertices on the given boundary curve and then triangulate the space between the previous curve and the new curve. We repeat this process, and stop when the number of vertices in the inner curve reduces to three or fewer vertices. The number of new interior closed curves will be in the order of the logarithm

of the number of points on the boundary of the input closed curve.

One iteration of the process is as follows: the given curve is projected onto a plane that best approximates the curve points. A new vertex is introduced for every alternate vertex on the boundary, and the new vertex is connected to its corresponding boundary vertex and its two neighbors. Two new vertices are connected to each other, if they have a common boundary neighbor. The sequence of new vertices form the new boundary curve. The above mentioned connectivity triangulates the space between the previous and the new curves. The initial coordinate of the new vertex is computed as the centroid of four points: the three boundary vertices and the centroid P of the points on the boundary of the original input curve. The point P is fixed and does not change over iterations. This approach to compute coordinates can be improved by taking into account the number of closed inner curves that will be computed.

Surface Fairing: Once topological structure for patch is generated, final geometric positions of the new vertices are computed to make the limit surface fair. For every internal vertex, in the order in which they were found in the topological triangulation step, we use its initially assigned coordinates and the tangent planes at its neighbors in the boundary, to find the final position of this new vertex. We take the line defined by the position of the internal vertex and its normal, and find the intersection of this line with each neighboring tangent plane. The median position of these intersection points is the new position of the vertex. Similar to the topological triangulation step, the surface fairing step is iterated to find the final positions of all the newly introduced vertices. Further iterations of this algorithm reduce the noise in the result.

Our approach to surface fitting to the network of curves provides a large number of internal vertices having degree five, except for those in the last layer of the triangulation. It is simple, has fast convergence, and interactive. It can handle models with any arbitrary genus and works for convex patches and those with low concavity.

5. Implementation and Results

We have tested our approach generating a number of simple models (see Figure 4), with normally six to ten curves, producing a similar number of patches. It took an average of about one minute for the users to sketch simple models using our system, and the automatic generation was executed at interactive rates on a 2.4 Ghz Pentium 4 processor with 512 MB RAM. Our system provides a simple interface (see Figure 5) where the user can draw lines, rotate the view and continue drawing until the desired shape is achieved. Basic utilities like picking, copying and moving curves, or indicating corner points, are part of the interface. The identification of surface patches is currently manual but will be automated

in future. In order to reduce the effect of the possible outliers in the surface fairing procedure, we damp the migration of vertices, only allowing displacements proportional to the length of surrounding edges. This way the relative movement of vertices is controlled by a user-defined parameter. The surface generated gives a coarse representation of the object and can be further refined using subdivision surfaces.

6. Conclusion

In this paper we have addressed the problem of generating a free form surface from a network of curves which are in turn generated using 2D sketches. The 3D space curve generation technique uses a single view information about a 2D curve to find the curvature minimizing 3D curve. We believe that the mathematical rigor used to explain the intuitive space curve generation would find its use in perceptual studies. The 3D surface generation algorithm using the network of curves is simple and efficient that is required for interactive applications like sketch based modeling. The generated surface might not be the final surface that the user intends to keep. Hence, once the initial surface is generated, users can modify the shape of the model interactively thus making the system analogous to a virtual sculpting tool.

Planned future work in our sketch-based free-form modeling system includes intelligent and automatic patch identification for surface generation, generating models having surface with boundary, using a non-iterative vertex location procedure during surface fairing, and improving the basic sketch interface providing more utilities to the user thereby making a simple, easy-to-use sketch-based free-form modeling system for generation of intuitive free-form objects.

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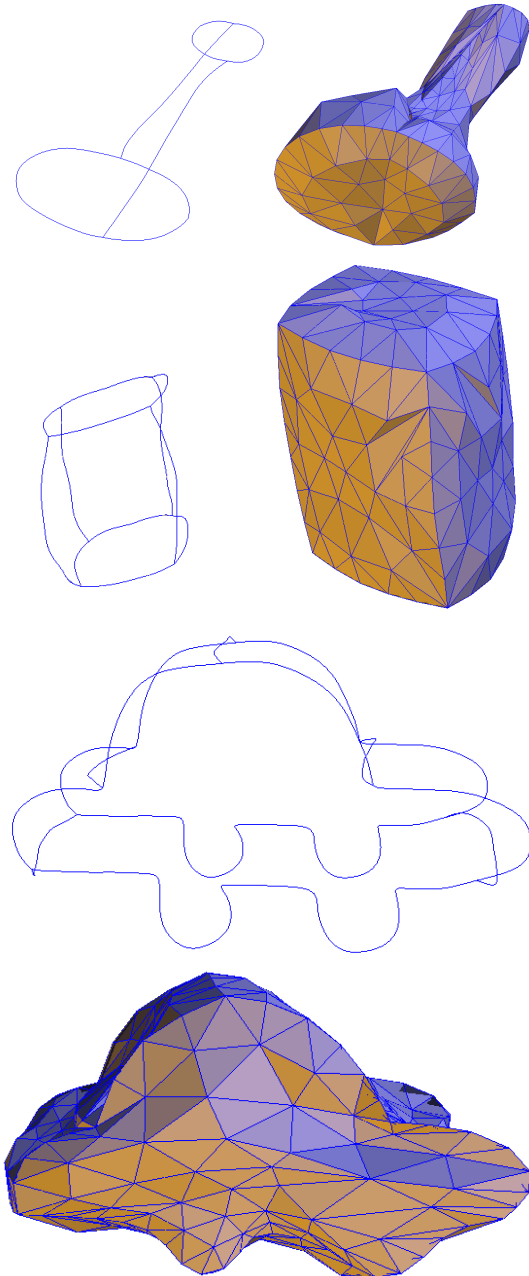


Figure 4: Simple models generated by our system along with the input sketch

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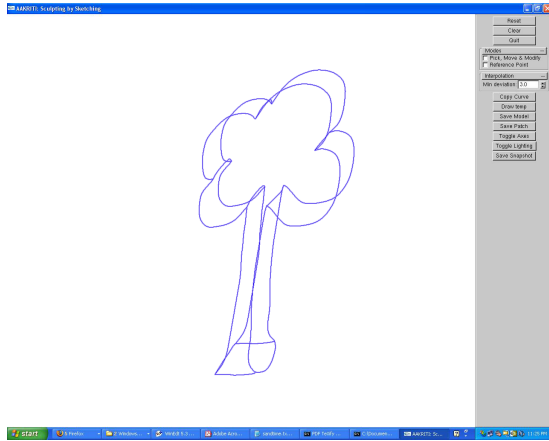


Figure 5: User interface showing one of the sketch inputs for generated models.

flowers. Technical Sketch at ACM SIGGRAPH, August 2004.

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Appendix: Proof of Theorem

Theorem 1: Given a 2D Bezier curve with a known depth range, the depth of the control points of curvature minimizing 3D Bezier curve is monotonically and equally spaced within the given depth range.

Proof (Sketch) For the sake of simplicity of mathematical proof, we assume that the given plane is parallel to the XY plane and the viewing direction is the Z axis. Let us also assume that the range denotes the depth value of the first and the last control point of the Bezier curve.

Let the control points of the 2D Bezier curve be denoted by $P_i = (x_i, y_i)$, $0 \leq i \leq n$, where n is the order of the curve. The Bezier curves are orthographic projection invariant. Hence the 3D curve will have the same number of control points as its 2D projection, and the control points of the 2D curve are actually the projection of the 3D curve control points. Since it is an orthographic projection, the control points of all the candidate 3D space curves will have the same x and y coordinates. Let the 3D control points be $Q_i = (x_i, y_i, z_i)$. The curve is,

$$Q(u) = \sum_{i=0}^n \frac{n!}{i!(n-i)!} Q_i u^i (1-u)^{n-i}$$

Differentiating, we get

$$Q'(u) = n \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} [Q_{i+1} - Q_i] u^i (1-u)^{(n-1-i)}$$

and the second derivative is given by,

$$Q''(u) = (n)(n-1) \sum_{i=0}^{n-2} \frac{(n-2)!}{i!(n-2-i)!} [Q_{i+2} - 2Q_{i+1} + Q_i] u^i (1-u)^{(n-2-i)}$$

The normal curvature κ is given by $\frac{|Q' \times Q''|}{|Q'|^3}$. Intuitively, to minimize the maximum curvature, we need to minimize the maximum value of Q'' and maximize the minimum value of Q' .

With respect to Q' , larger the differences in depth values of the control points, larger the denominator and hence the minimum value has no maxima. Hence, minimizing curvature depends only on the magnitude of Q'' . This can also be seen from the fact that since curvature is an implicit characteristic of a curve, for an equivalent arc-length parameterization, the curvature is controlled by its second derivative. Based on the above argument, our goal is to minimize the magnitude of the 3D vectors $(Q_{i+2} - 2Q_{i+1} + Q_i)$ for all i . Since the x and y coordinates of Q_i are fixed by the input 2D Bezier control points, and the z coordinates of the first and last control points are fixed by the given depth range, the only free parameters are the z coordinates of the intermediate control points Q_i , $1 \leq i \leq (n-1)$. Minimizing $|z_{i+2} - 2z_{i+1} + z_i|$ would minimize the magnitude of the required vector. Since the z_0 and z_n are known, setting $z_{i+2} - 2z_{i+1} + z_i = 0$ for all i provides a set of recurrence equations. The solution for this set of equations is

$$z_i = z_0 + \frac{z_n - z_0}{n} i.$$

□