Constraint Satisfaction Problems (CSPs)

Introduction and Backtracking Search

This lecture topic (two lectures)
Chapter 6.1 – 6.4, except 6.3.3

Next lecture topic (two lectures)
Chapter 7.1 – 7.5

(Please read lecture topic material before and after each lecture on that topic)

Outline

- What is a CSP?
- Backtracking Search for CSP
- Variable selection (ordering)
  - Minimum Remaining Values (MRV) heuristic
  - Degree Heuristic
- Value selection (ordering)
  - Least Constraining Value (LCV) heuristic

You Will Be Expected to Know

- Basic definitions (section 6.1)
- Backtracking search (6.3)
- Variable ordering or selection (6.3.1)
  - minimum-remaining values
  - degree heuristic
- Value ordering or selection (6.3.1)
  - least-constraining-value

Constraint Satisfaction Problems

- What is a CSP?
  - Finite set of variables $X_1, X_2, \ldots, X_n$
  - Nonempty domain of possible values for each variable $D_1, D_2, \ldots, D_n$
  - Finite set of constraints $C_1, C_2, \ldots, C_m$
    - Each constraint C_i limits the values that variables can take,
    - e.g., $X_1 \neq X_2$
  - Each constraint C_i is a pair <scope, relation>
    - Scope = Tuple of variables that participate in the constraint.
    - Relation = List of allowed combinations of variable values.
      May be an explicit list of allowed combinations.
      May be an abstract relation allowing membership testing and listing.

- CSP benefits
  - Standard representation pattern
  - Generic goal and successor functions
  - Generic heuristics (no domain specific expertise).

Sudoku as a Constraint Satisfaction Problem (CSP)

	1	2	3	4	5	6	7	8	9
A		6		1		4		5	
B			8	3		5	6		
C	2								1
D	8			4		7			6
E			6				3		
F	7			9		1			4
G	5								2
H			7	2		6	9		
I		4		5		8		7	

- Variables: 81 variables
  - A1, A2, A3, ..., I7, I8, I9
  - Letters index rows, top to bottom
  - Digits index columns, left to right
- Domains: The nine positive digits
  - A1 $\in$ {1, 2, 3, 4, 5, 6, 7, 8, 9}
  - Etc.
- Constraints: 27 *Alldiff* constraints
  - *Alldiff*(A1, A2, A3, A4, A5, A6, A7, A8, A9)
  - Etc.

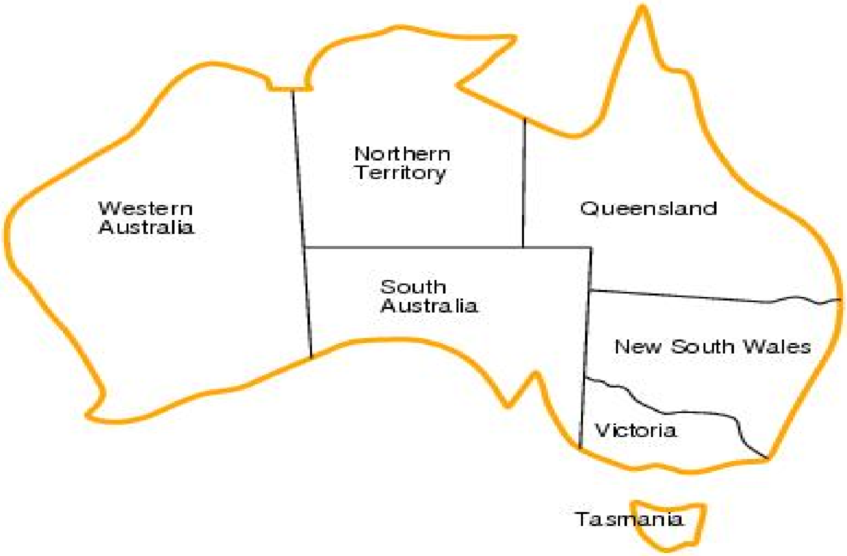
Random Binary CSP (adapted from http://www.unitime.org/csp.php)

- A random binary CSP is defined by a four-tuple (n, d, p1, p2)
  - n = the number of variables.
  - d = the domain size of each variable.
  - p1 = probability a constraint exists between two variables.
  - p2 = probability a pair of values in the domains of two variables connected by a constraint is incompatible.
    - Note that R&N lists compatible pairs of values instead.
    - Equivalent formulations; just take the set complement.
- (n, d, p1, p2) are used to generate randomly the binary constraints among the variables.
- The so called model B of Random CSP (n, d, n1, n2)
  - n1 = p1n(n-1)/2 pairs of variables are randomly and uniformly selected and binary constraints are posted between them.
  - For each constraint, n2 = p2d^2 randomly and uniformly selected pairs of values are picked as incompatible.
- The random CSP as an optimization problem (minCSP).
  - Goal is to minimize the total sum of values for all variables.

CSPs --- what is a solution?

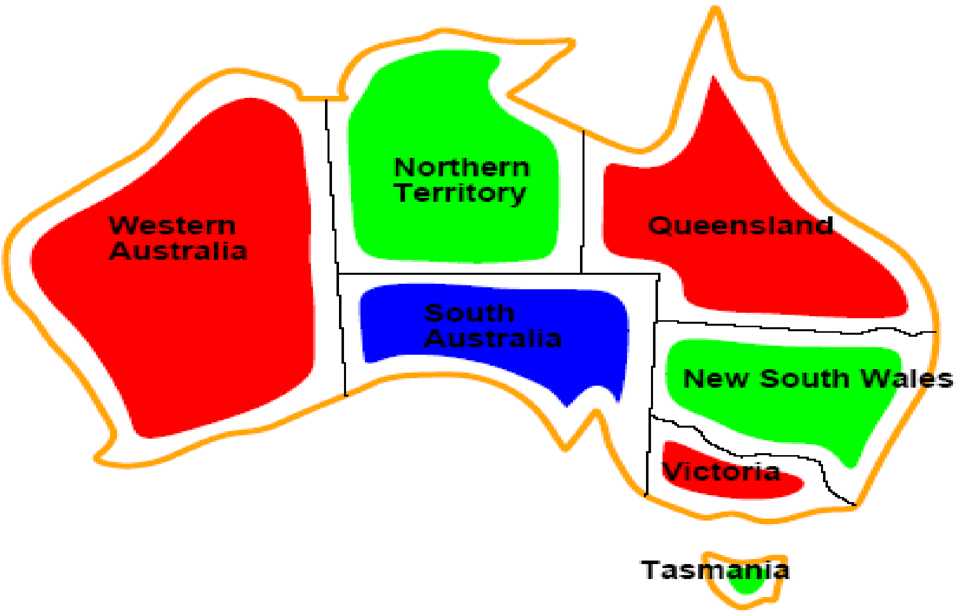
- A *state* is an *assignment* of values to some or all variables.
  - An assignment is *complete* when every variable has a value.
  - An assignment is *partial* when some variables have no values.
- ***Consistent assignment***
  - assignment does not violate the constraints
- A ***solution*** to a CSP is a complete and consistent assignment.
- Some CSPs require a solution that maximizes an *objective function*.
- Examples of Applications:
  - Scheduling the time of observations on the Hubble Space Telescope
  - Airline schedules
  - Cryptography
  - Computer vision -> image interpretation
  - Scheduling your MS or PhD thesis exam ☺

CSP example: map coloring



- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: $D_i = \{red, green, blue\}$
- Constraints:adjacent regions must have different colors.
  - E.g. $WA \neq NT$

CSP example: map coloring



- Solutions are assignments satisfying all constraints, e.g.
  {WA=red,NT=green,Q=red,NSW=green,V=red,SA=blue,T=green}

Graph coloring

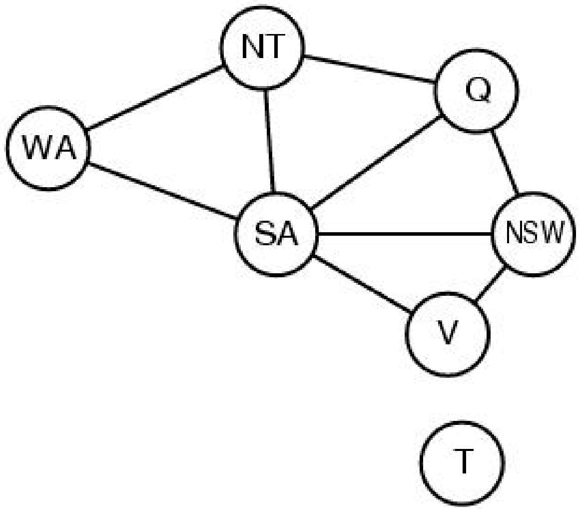
- More general problem than map coloring
- Planar graph = graph in the 2d-plane with no edge crossings
- Guthrie's conjecture (1852)

  Every planar graph can be colored with 4 colors or less

  - Proved (using a computer) in 1977 (Appel and Haken)

Constraint graphs

- Constraint graph:
  - nodes are variables
  - arcs are binary constraints



- Graph can be used to simplify search
  e.g. Tasmania is an independent subproblem

(will return to graph structure later)

Varieties of CSPs

- Discrete variables
  - Finite domains; size *d* $\Rightarrow O(d^n)$ complete assignments.
    - E.g. Boolean CSPs: Boolean satisfiability (NP-complete).
  - Infinite domains (integers, strings, etc.)
    - E.g. job scheduling, variables are start/end days for each job
    - Need a constraint language e.g $StartJob_1 + 5 \leq StartJob_3$.
    - Infinitely many solutions
    - Linear constraints: solvable
    - Nonlinear: no general algorithm
- Continuous variables
  - e.g. building an airline schedule or class schedule.
  - Linear constraints solvable in polynomial time by LP methods.

Varieties of constraints

- Unary constraints involve a single variable.
  - e.g. $SA \neq green$
- Binary constraints involve pairs of variables.
  - e.g. $SA \neq WA$
- Higher-order constraints involve 3 or more variables.
  - Professors A, B,and C cannot be on a committee together
  - Can always be represented by multiple binary constraints
- Preference (soft constraints)
  - e.g. *red* is better than *green* often can be represented by a cost for each variable assignment
  - combination of optimization with CSPs

CSPs Only Need Binary Constraints!!

- Unary constraints: Just delete values from variable's domain.
- Higher order (3 variables or more): reduce to binary constraints.
- Simple example:
  - Three example variables, X, Y, Z.
  - Domains Dx={1,2,3}, Dy={1,2,3}, Dz={1,2,3}.
  - Constraint C[X,Y,Z] = {***X+Y=Z***} = {(1,1,2), (1,2,3), (2,1,3)}.
  - Plus many other variables and constraints elsewhere in the CSP.

  - Create a new variable, W, taking values as triples (3-tuples).
  - Domain of W is Dw = {(1,1,2), (1,2,3), (2,1,3)}.
    - Dw is exactly the tuples that satisfy the higher order constraint.
  - Create three new constraints:
    - C[X,W] = { [1, (1,1,2)], [1, (1,2,3)], [2, (2,1,3)] }.
    - C[Y,W] = { [1, (1,1,2)], [2, (1,2,3)], [1, (2,1,3)] }.
    - C[Z,W] = { [2, (1,1,2)], [3, (1,2,3)], [3, (2,1,3)] }.
  - Other constraints elsewhere involving X, Y, or Z are unaffected.

CSP Example: Cryptharithmetic puzzle

$$
\begin{array}{cccc}
 & T & W & O \\
+ & T & W & O \\
\hline
F & O & U & R
\end{array}
$$

Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Constraints
alldiff(F, T, U, W, R, O)
$O + O = R + 10 \cdot X_1$, etc.

CSP Example: Cryptharithmetic puzzle

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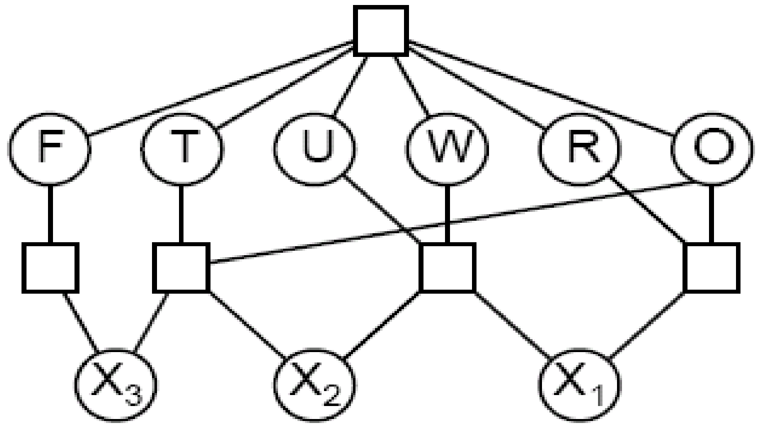

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alldiff(F, T, U, W, R, O)

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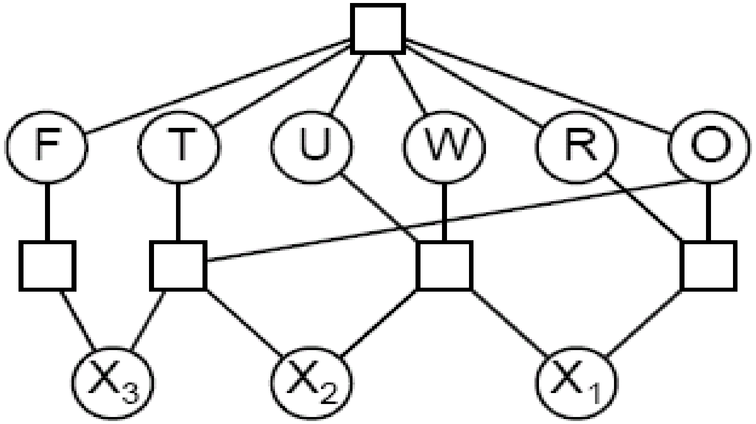
CSP Example: Cryptharithmetic puzzle

```
   T W O
+  T W O
---------
 F O U R
```



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Constraints

alldiff(F,T,U,W,R,O)

$O+O=R+10\cdot X_1$, etc.

A Solution:

F=1, T=7, U=6, W=3, R=8, O=4, X1=0, X2=0, X3=1

```
   7 3 4
 + 7 3 4
 -------
 1 4 6 8
```

CSP Example: Cryptharithmetic puzzle

- Try it yourself at home:

```
    S E N D
  + M O R E
  ---------
  M O N E Y
```

- (A frequent request from college students to parents!)

CSP as a standard search problem

- A CSP can easily be expressed as a standard search problem.

- Incremental formulation

  - *Initial State*: the empty assignment {}

  - *Actions*: Assign a value to an unassigned variable provided that it does not violate a constraint

  - *Goal test*: the current assignment is complete
    (by construction it is consistent)

  - *Path cost*: constant cost for every step (not really relevant)

- Can also use complete-state formulation
  - Local search techniques (Chapter 4) tend to work well

CSP as a standard search problem

- Solution is found at depth *n* (if there are *n* variables).

- Consider using BFS
  - Branching factor *b* at the top level is *nd*
  - At next level is (n-1)d
  - ....

- end up with $n!d^n$ leaves even though there are only d^n complete assignments!

Commutativity

- CSPs are commutative.
  - The order of any given set of actions has no effect on the outcome.
  - Example: choose colors for Australian territories one at a time
    - [WA=red then NT=green] same as [NT=green then WA=red]

- All CSP search algorithms can generate successors by considering assignments for only a single variable at each node in the search tree

  $\Rightarrow$ there are d^n leaves

(will need to figure out later which variable to assign a value to at each node)

Backtracking search

- Similar to Depth-first search, generating children one at a time.
- Chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.
- Uninformed algorithm
  - No good general performance

Backtracking search (Figure 6.5)

```
function BACKTRACKING-SEARCH(csp) return a solution or failure
    return RECURSIVE-BACKTRACKING({ } , csp)

function RECURSIVE-BACKTRACKING(assignment, csp) return a solution or failure
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```
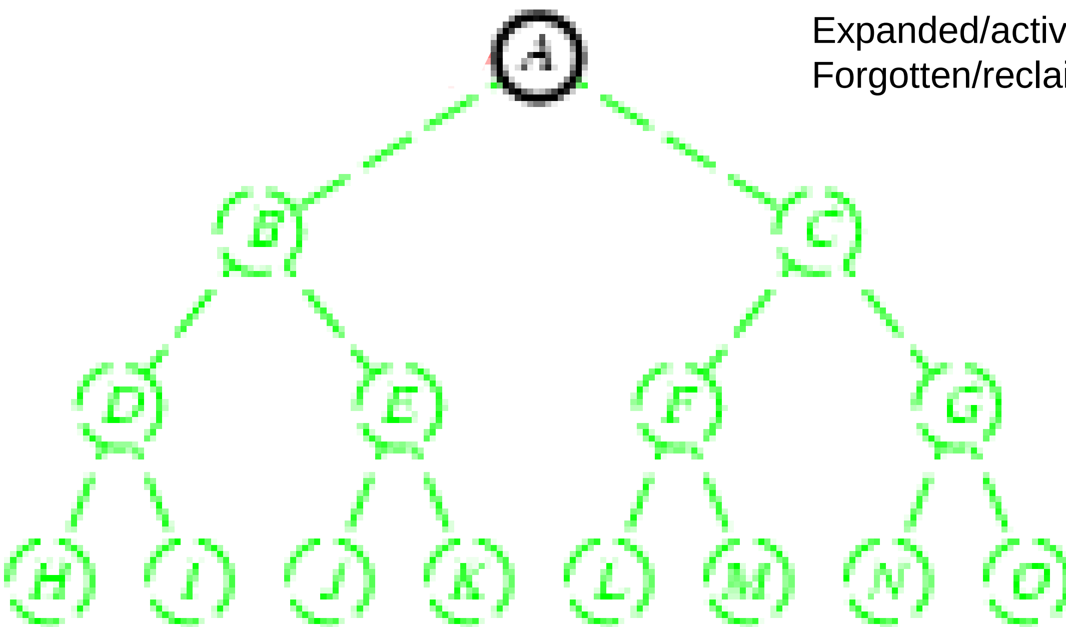

Backtracking search

- Expand *deepest* unexpanded node
- Generate ***only one*** child at a time.
- *Goal-Test* when inserted.
  - For CSP, Goal-test at bottom

Future= green dotted circles
Frontier=white nodes
Expanded/active=gray nodes
Forgotten/reclaimed= black nodes



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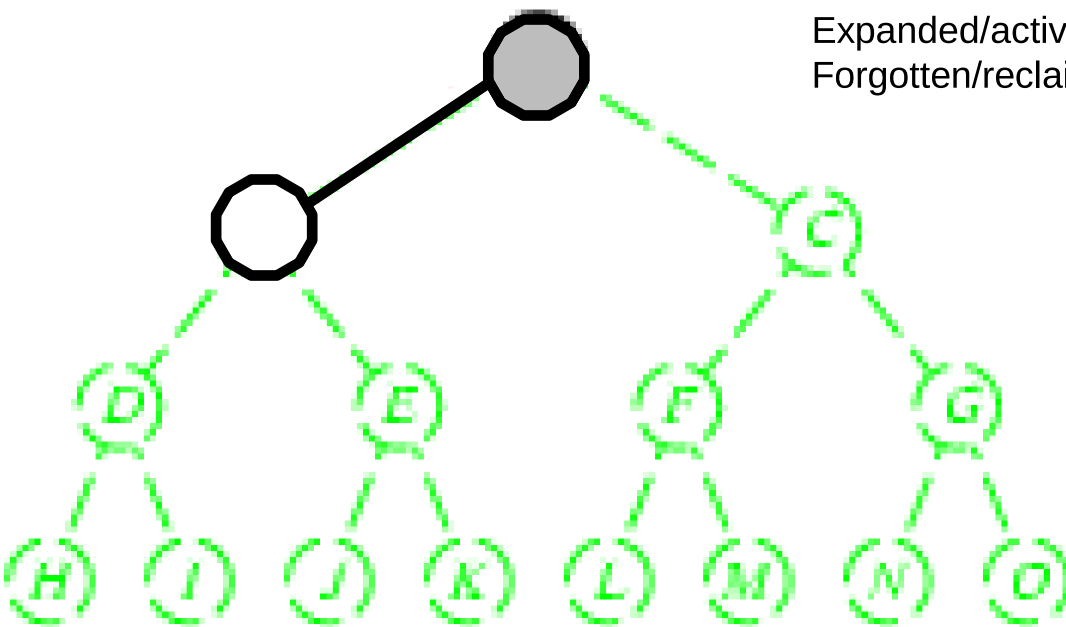

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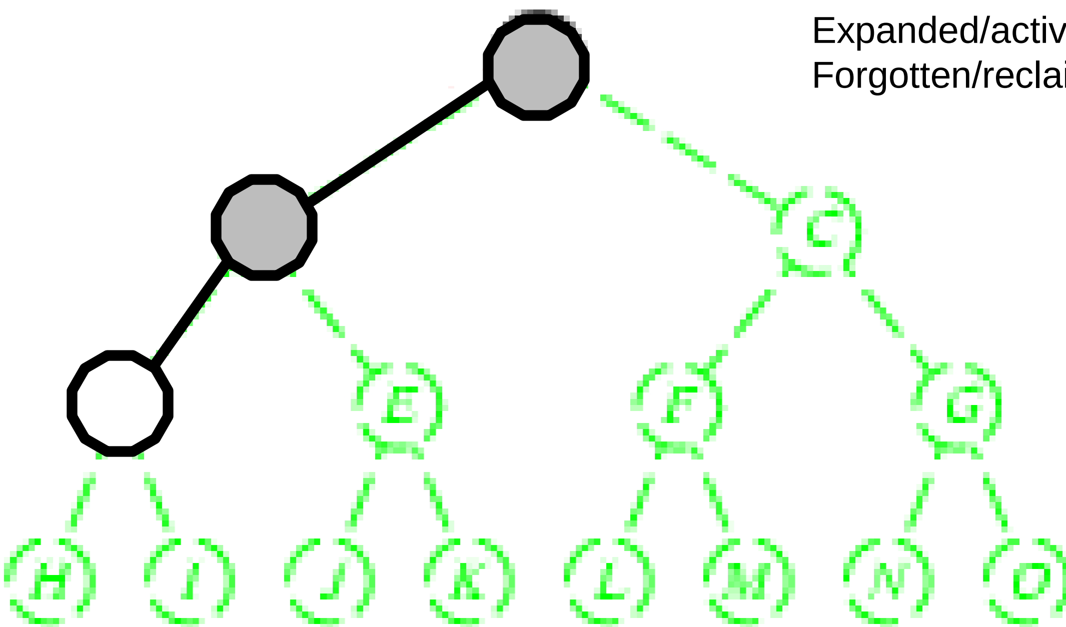

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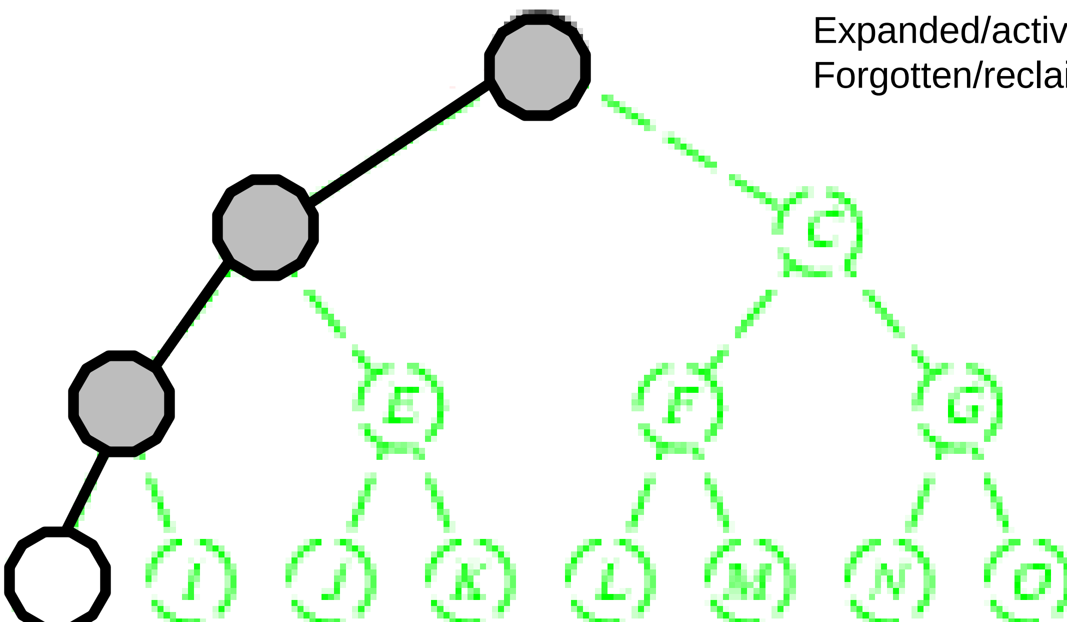

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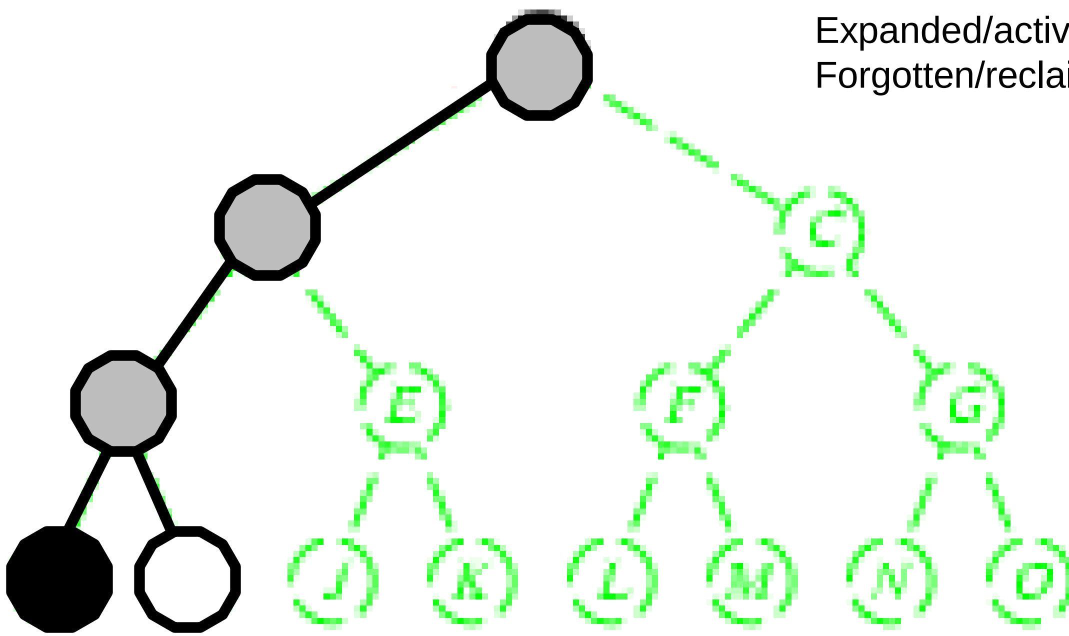

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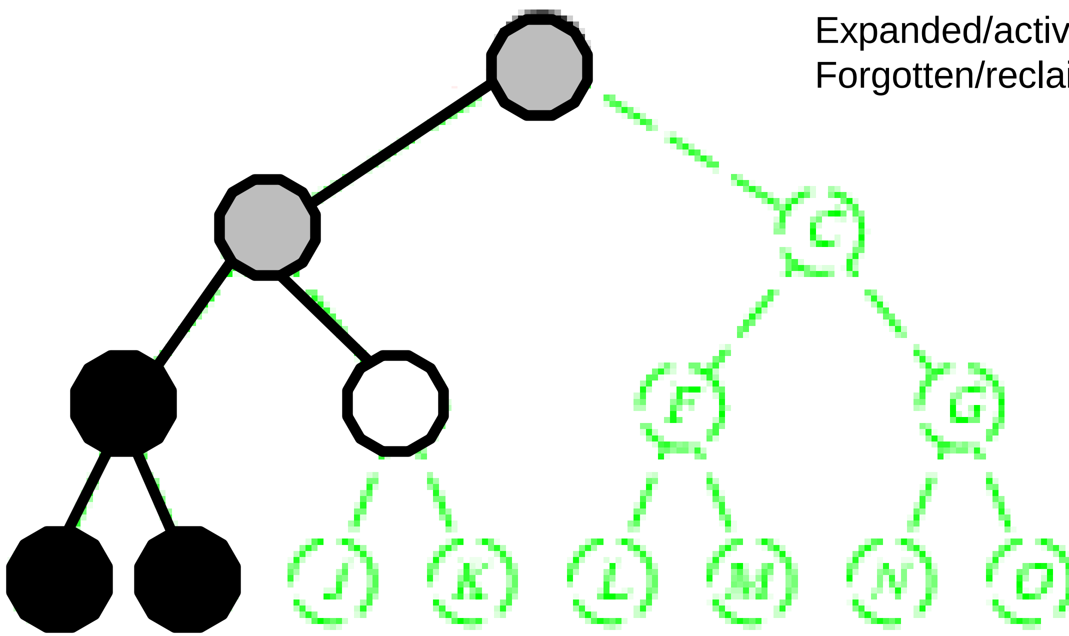

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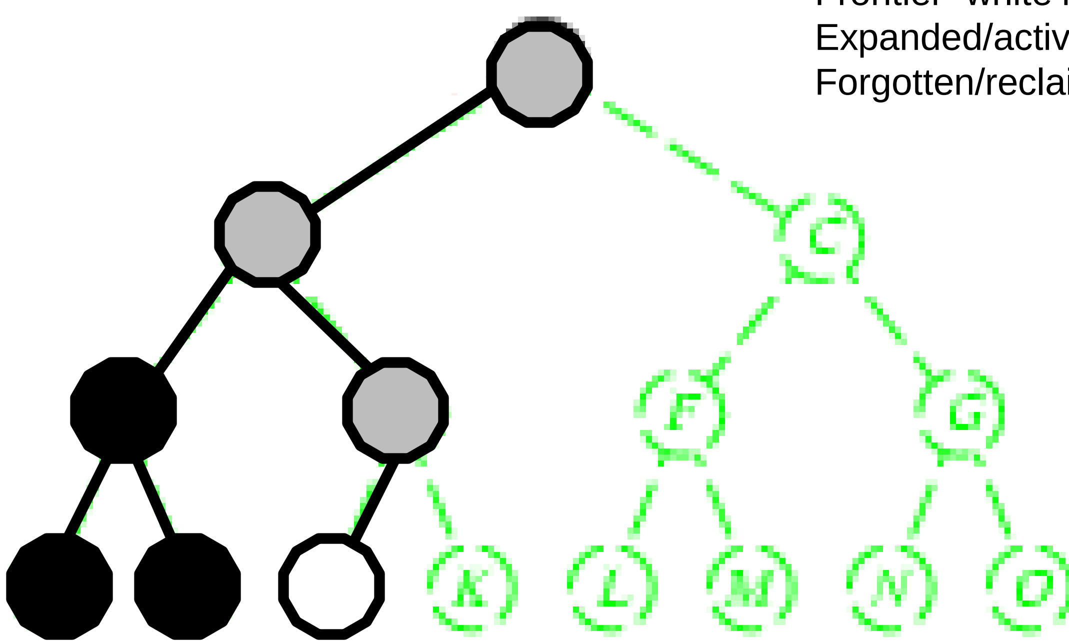

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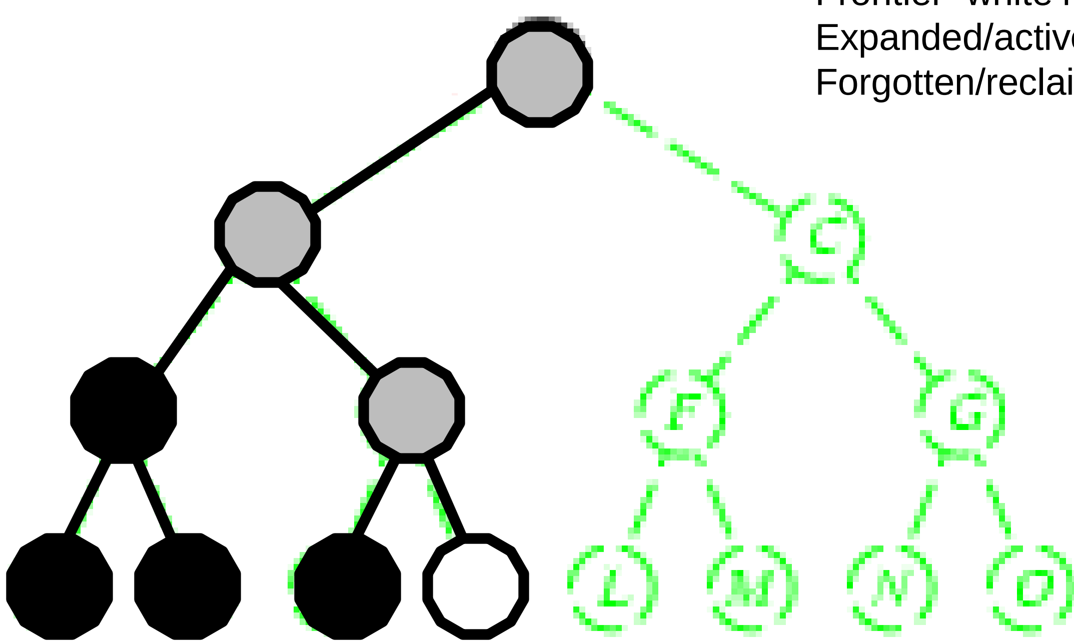

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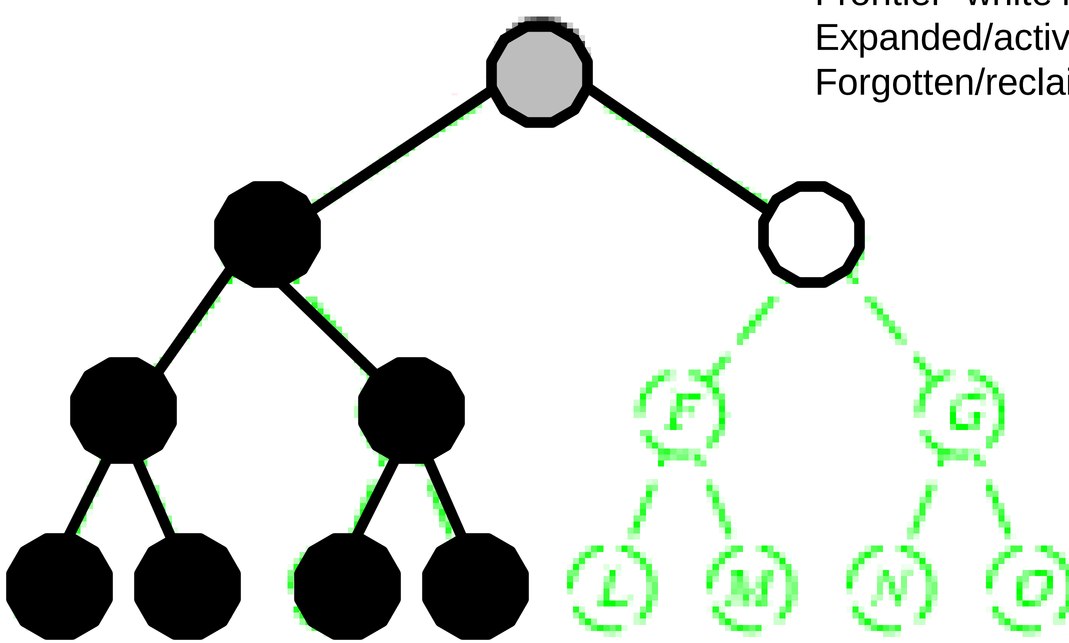

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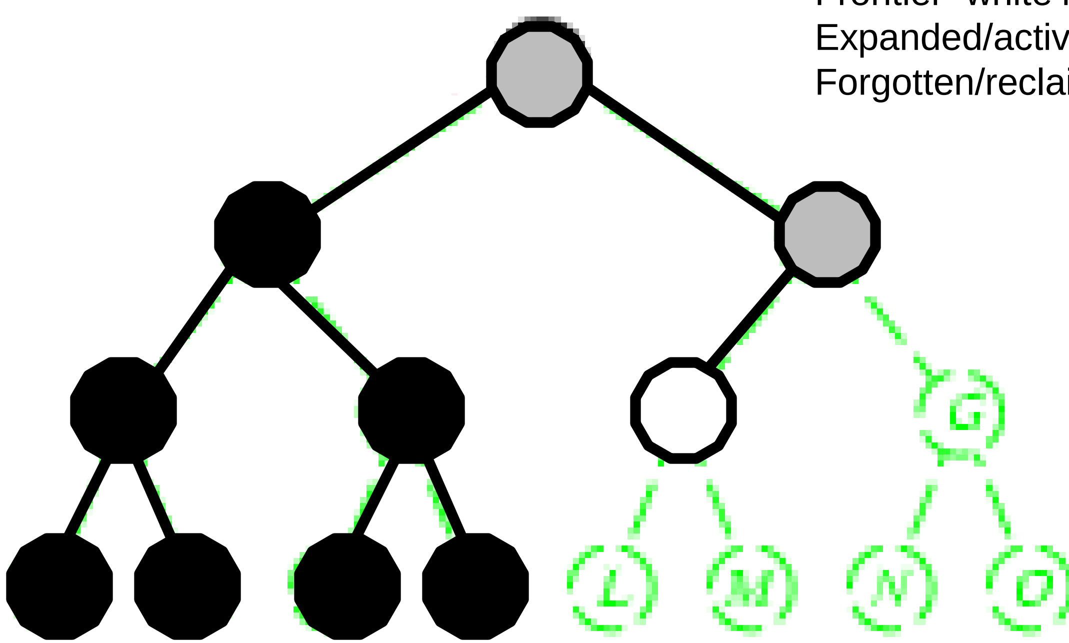

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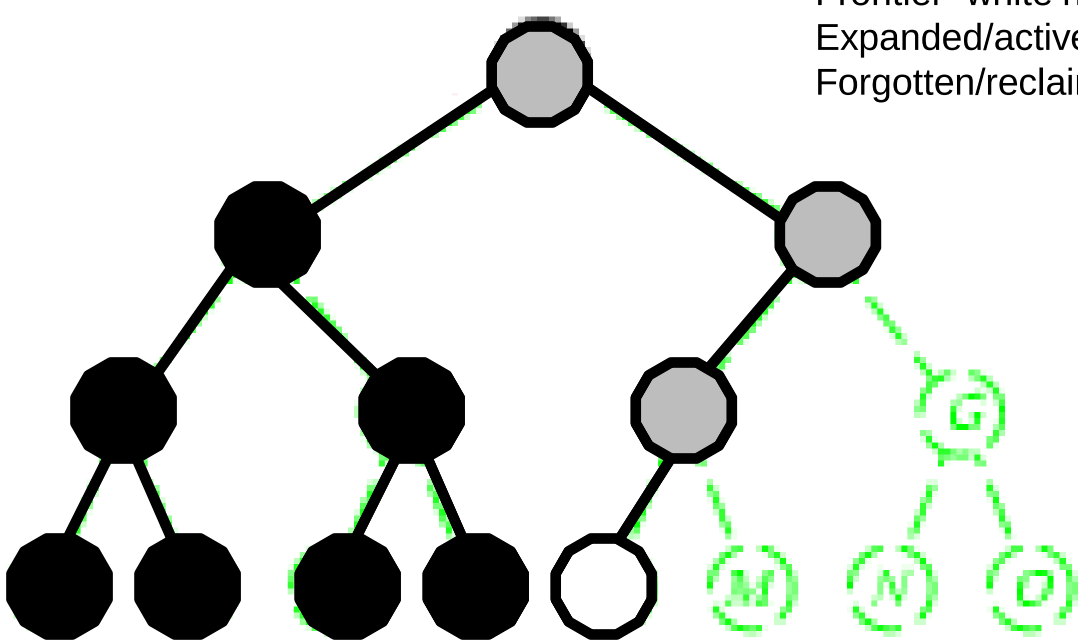

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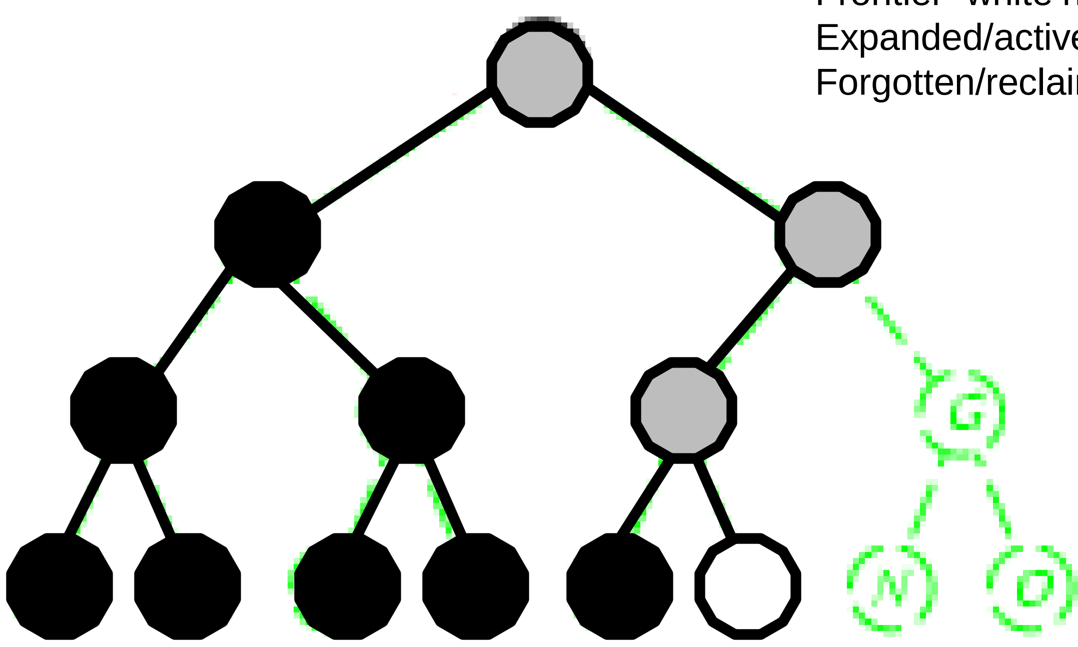

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Improving CSP efficiency

- Previous improvements on uninformed search
  $\rightarrow$ introduce heuristics

- For CSPS, general-purpose methods can give large gains in speed, e.g.,
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

Note: CSPs are somewhat generic in their formulation, and so the heuristics are more general compared to methods in Chapter 4

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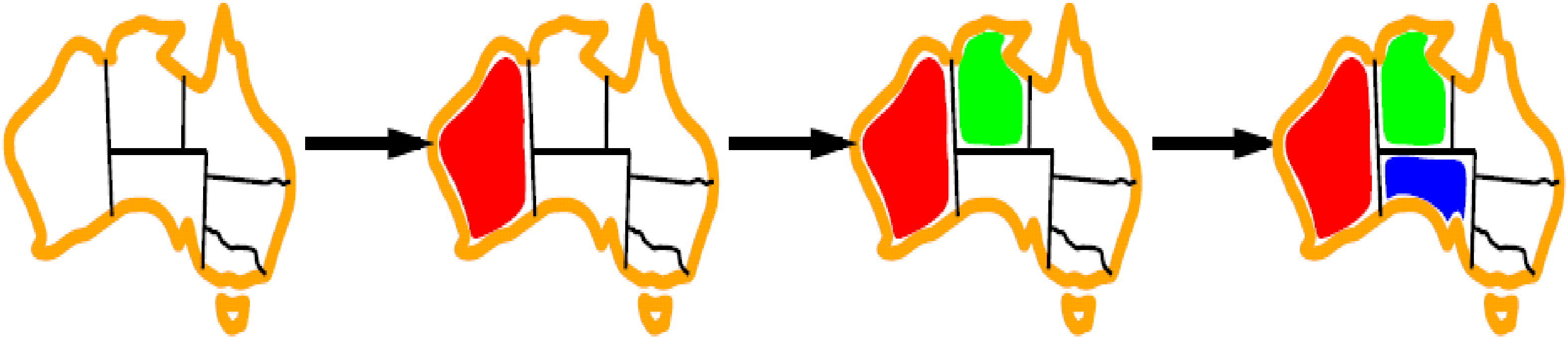
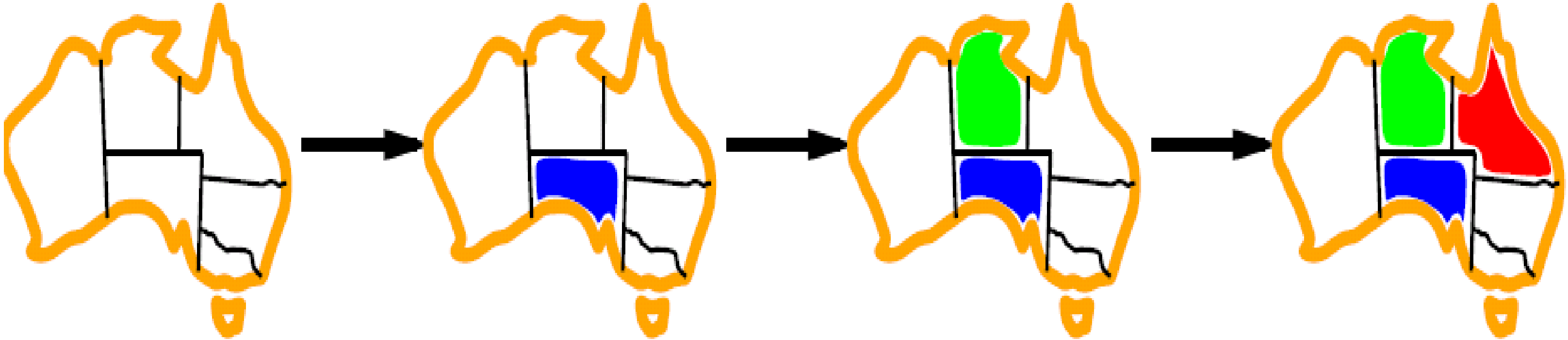
Minimum remaining values (MRV) for next variable



var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)

- A.k.a. most constrained variable heuristic
- *Heuristic Rule*: choose variable with the fewest legal moves
  - e.g., will immediately detect failure if X has no legal values

Degree heuristic for next variable



- *Heuristic Rule*: select variable that is involved in the largest number of constraints on other unassigned variables.
- Degree heuristic can be useful as a tie breaker after MRV.
- *In what order should a variable's values be tried?*

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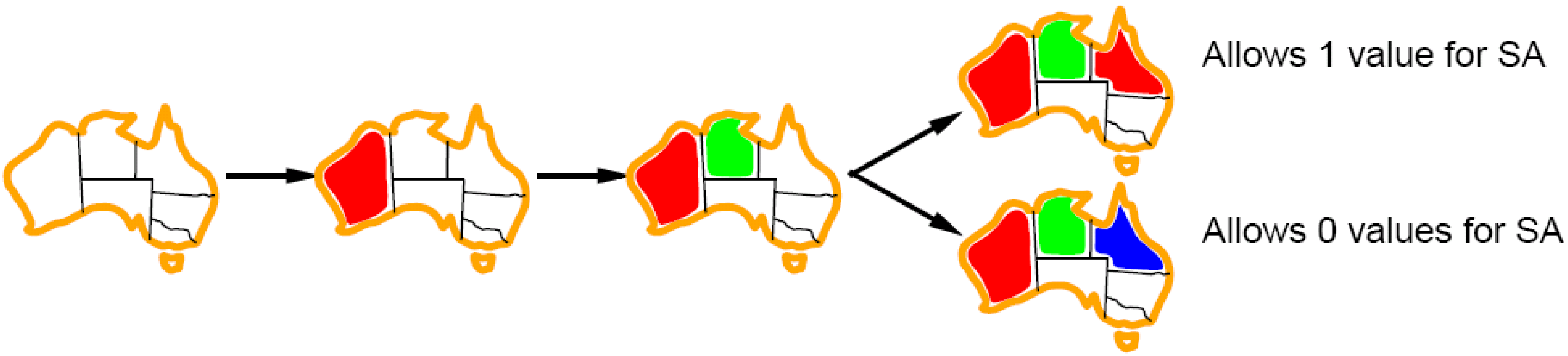
Least constraining value (LCV) for next value



- Least constraining value heuristic

- Heuristic Rule: given a variable choose the least constraining value
  - leaves the maximum flexibility for subsequent variable assignments

Minimum remaining values (MRV) vs. Least constraining value (LCV)

- Why do we want the MRV (minimum values, most constraining) for variable selection --- but the LCV (maximum values, least constraining) for value selection?

- <u>Isn't there a contradiction here?</u>

- MRV for variable selection to reduces the branching factor.
  - Smaller branching factors lead to faster search.
  - Hopefully, when we get to variables with currently many values, constraint propagation (next lecture) will have removed some of their values and they'll have small branching factors by then too.
- LCV for value selection increases the chance of early success.
  - If we are going to fail at this node, then we have to examine every value anyway, and their order makes no difference at all.
  - If we are going to succeed, then the earlier we succeed the sooner we can stop searching, so we want to succeed early.
  - LCV rules out the fewest possible solutions below this node, so we have the most chances for early success.

Summary

- CSPs
  - special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values

- Backtracking=depth-first search with one variable assigned per node

- Heuristics
  - Variable ordering and value selection heuristics help significantly

- Variable ordering (selection) heuristics
  - Choose variable with Minimum Remaining Values (MRV)
  - Degree Heuristic --- break ties after applying MRV

- Value ordering (selection) heuristic
  - Choose Least Constraining Value