Constraint Satisfaction Problems (CSPs)

Constraint Propagation and Local Search

This lecture topic (two lectures)
Chapter 6.1 – 6.4, except 6.3.3

Next lecture topic (two lectures)
Chapter 7.1 – 7.5

(Please read lecture topic material before and after each lecture on that topic)
Outline

• Constraint Propagation for CSP

• Forward Checking
  – Book-keeping can be tricky when backtracking

• Node / Arc / Path Consistency, K-Consistency
  – AC-3

• Global Constraints (any number of variables)
  – Special-purpose code often much more efficient

• Local search for CSPs
  – Min-Conflicts heuristic

• (Removed) Problem structure and decomposition
You Will Be Expected to Know

- Node consistency, arc consistency, path consistency (6.2)
- Forward checking (6.3.2)
- Local search for CSPs: min-conflict heuristic (6.4)
function BACKTRACKING-SEARCH(csp) return a solution or failure
  return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) return a solution or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment according to CONSTRAINTS[csp] then
      add {var=value} to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove {var=value} from assignment
  return failure
Improving CSP efficiency

- Previous improvements on uninformed search → introduce heuristics

- For CSPS, general-purpose methods can give large gains in speed, e.g.,
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

Note: CSPs are somewhat generic in their formulation, and so the heuristics are more general compared to methods in Chapter 4
Minimum remaining values (MRV)

\[ \text{var} \leftarrow \text{SELECT-UNASSIGNED-VARIABLE}(\text{VARIABLES}[csp], \text{assignment}, csp) \]

- A.k.a. most constrained variable heuristic

- **Heuristic Rule**: choose variable with the fewest legal moves
  - e.g., will immediately detect failure if \( X \) has no legal values
Degree heuristic for the initial variable

- **Heuristic Rule**: select variable that is involved in the largest number of constraints on other unassigned variables.
- Degree heuristic can be useful as a tie breaker.
- *In what order should a variable’s values be tried?*
Least constraining value for value-ordering

- Least constraining value heuristic

- Heuristic Rule: given a variable choose the least constraining value
  - leaves the maximum flexibility for subsequent variable assignments

Allows 1 value for SA

Allows 0 values for SA
Forward checking

- Can we detect inevitable failure early?
  - And avoid it later?

- *Forward checking idea*: keep track of remaining legal values for unassigned variables.

- Terminate search when any variable has no legal values.
Forward checking

- Assign \{WA=red\}

- Effects on other variables connected by constraints to WA
  - \textit{NT can no longer be red}
  - \textit{SA can no longer be red}
**Forward checking**

- Assign \(Q=\text{green}\)

- Effects on other variables connected by constraints with WA
  - \(NT\) can no longer be green
  - \(NSW\) can no longer be green
  - \(SA\) can no longer be green

- **MRV heuristic** would automatically select NT or SA next
Forward checking

- If \( V \) is assigned \textit{blue}

- Effects on other variables connected by constraints with WA
  - \textit{NSW can no longer be blue}
  - \textit{SA is empty}

- FC has detected that partial assignment is \textit{inconsistent} with the constraints and backtracking can occur.
Example: 4-Queens Problem
Example: 4-Queens Problem

Red = value is assigned to variable
Example: 4-Queens Problem

Red = value is assigned to variable
Example: 4-Queens Problem

- X1 Level:
  - Deleted:
    - \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

- (Please note: As always in computer science, there are many different ways to implement anything. The book-keeping method shown here was chosen because it is easy to present and understand visually. It is not necessarily the most efficient way to implement the book-keeping in a computer. Your job as an algorithm designer is to think long and hard about your problem, then devise an efficient implementation.)

- One more efficient equivalent possible alternative (of many):
  - Deleted:
    - \{ (X2:1,2) (X3:1,3) (X4:1,4) \}
Example: 4-Queens Problem

X1 \{1,2,3,4\}

X2 \{ , ,3,4\}

X3 \{ ,2, ,4\}

X4 \{ ,2,3, ,\}

Red = value is assigned to variable
Example: 4-Queens Problem

Red = value is assigned to variable
Example: 4-Queens Problem

Red = value is assigned to variable
Example: 4-Queens Problem

- X1 Level:
  - Deleted:
    - \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

- X2 Level:
  - Deleted:
    - \{ (X3,2) (X3,4) (X4,3) \}

(Please note: Of course, we could have failed as soon as we deleted \{ (X3,2) (X3,4) \}. There was no need to continue to delete (X4,3), because we already had established that the domain of X3 was null, and so we already knew that this branch was futile and we were going to fail anyway. The book-keeping method shown here was chosen because it is easy to present and understand visually. It is not necessarily the most efficient way to implement the book-keeping in a computer. Your job as an algorithm designer is to think long and hard about your problem, then devise an efficient implementation.)
Example: 4-Queens Problem

Red = value is assigned to variable
Example: 4-Queens Problem

• X1 Level:
  – Deleted:
    • \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

• X2 Level:
  – **FAIL at X2=3.**
  – **Restore:**
    • \{ (X3,2) (X3,4) (X4,3) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

Red = value is assigned to variable
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Red = value is assigned to variable
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Example: 4-Queens Problem

- **X1 Level:**
  - Deleted:
    - \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

- **X2 Level:**
  - Deleted:
    - \{ (X3,4) (X4,2) \}
Example: 4-Queens Problem

Red = value is assigned to variable
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Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

• X1 Level:
  – Deleted:
  • \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

• X2 Level:
  – Deleted:
  • \{ (X3,4) (X4,2) \}

• X3 Level:
  – Deleted:
  • \{ (X4,3) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

- X1 Level:
  - Deleted:
    - \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

- X2 Level:
  - Deleted:
    - \{ (X3,4) (X4,2) \}

- X3 Level:
  - Fail at X3=2.
  - Restore:
    - \{ (X4,3) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

• X1 Level:
  – Deleted:
    • \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}

• X2 Level:
  – Fail at X2=4.
  – Restore:
    • \{ (X3,4) (X4,2) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

- X1 Level:
  - Fail at X1=1.
  - Restore:
    - \{ (X2,1) (X2,2) (X3,1) (X3,3) (X4,1) (X4,4) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

• X1 Level:
  – **Deleted:**
    • { (X2,1) (X2,2) (X2,3) (X3,2) (X3,4) (X4,2) }
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

Red = value is assigned to variable
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Example: 4-Queens Problem

Red = value is assigned to variable
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- X1 Level:
  - Deleted:
    - \{ (X2,1) (X2,2) (X2,3) (X3,2) (X3,4) (X4,2) \}

- X2 Level:
  - Deleted:
    - \{ (X3,3) (X4,4) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

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Example: 4-Queens Problem

- **X1 Level:**
  - Deleted:
    - \{ (X2,1) (X2,2) (X2,3) (X3,2) (X3,4) (X4,2) \}

- **X2 Level:**
  - Deleted:
    - \{ (X3,3) (X4,4) \}

- **X3 Level:**
  - **Deleted:**
    - \{ (X4,1) \}
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
Example: 4-Queens Problem

Red = value is assigned to variable
X = value led to failure
### Comparison of CSP algorithms on different problems

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Median number of consistency checks over 5 runs to solve problem

Parentheses -> no solution found

USA: 4 coloring
n-queens: n = 2 to 50
Zebra: see exercise 5.13
Constraint propagation

- Solving CSPs with combination of heuristics plus forward checking is more efficient than either approach alone

- FC checking does not detect all failures.
  - E.g., NT and SA cannot be blue
Constraint propagation

- Techniques like CP and FC are in effect eliminating parts of the search space
  - Somewhat complementary to search

- Constraint propagation goes further than FC by repeatedly enforcing constraints locally
  - Needs to be faster than actually searching to be effective

- Arc-consistency (AC) is a systematic procedure for constraint propagation
Arc consistency

- An Arc $X \rightarrow Y$ is consistent if for every value $x$ of $X$ there is some value $y$ consistent with $x$ (note that this is a directed property)

- Consider state of search after WA and Q are assigned:

$SA \rightarrow NSW$ is consistent if $SA=blue$ and $NSW=red$
Arc consistency

- $X \rightarrow Y$ is consistent if
  for every value $x$ of $X$ there is some value $y$ consistent with $x$

- $NSW \rightarrow SA$ is consistent if
  $NSW=\text{red}$ and $SA=\text{blue}$
  $NSW=\text{blue}$ and $SA=???
Arc consistency

- Can enforce arc-consistency:
  Arc can be made consistent by removing *blue* from *NSW*

- Continue to propagate constraints....
  - Check $V \rightarrow NSW$
  - Not consistent for $V = red$
  - Remove red from $V$
Arc consistency

- Continue to propagate constraints....

- \( SA \rightarrow NT \) is not consistent
  - and cannot be made consistent

- Arc consistency detects failure earlier than FC
Arc consistency checking

• Can be run as a preprocessor or after each assignment
  – Or as preprocessing before search starts

• AC must be run repeatedly until no inconsistency remains

• Trade-off
  – Requires some overhead to do, but generally more effective than direct search
  – In effect it can eliminate large (inconsistent) parts of the state space more effectively than search can

• Need a systematic method for arc-checking
  – If $X$ loses a value, neighbors of $X$ need to be rechecked:
    i.e. incoming arcs can become inconsistent again (outgoing arcs will stay consistent).
Arc consistency algorithm (AC-3)

**function** AC-3(csp) **returns** false if inconsistency found, else true, may reduce csp domains

**inputs:** csp, a binary CSP with variables \{X_1, X_2, ..., X_n\}

**local variables:** queue, a queue of arcs, initially all the arcs in csp

/* initial queue must contain both (X_i, X_j) and (X_j, X_i) */

**while** queue is not empty **do**

(X_i, X_j) ← REMOVE-FIRST(queue)

**if** REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**

**if** size of \(D_i\) = 0 **then return** false

**for each** X_k in NEIGHBORS[X_i] − \{X_j\} **do**

add (X_k, X_i) to queue if not already there

**return** true

**function** REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff we delete a value from the domain of X_i

removed ← false

**for each** x in DOMAIN[X_i] **do**

**if** no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraints between X_i and X_j

**then delete** x from DOMAIN[X_i]; removed ← true

**return** removed

(from Mackworth, 1977)
Complexity of AC-3

- A binary CSP has at most $n^2$ arcs

- Each arc can be inserted in the queue $d$ times (worst case)
  - $(X, Y)$: only $d$ values of $X$ to delete

- Consistency of an arc can be checked in $O(d^2)$ time

- Complexity is $O(n^2 \cdot d^3)$

- Although substantially more expensive than Forward Checking, Arc Consistency is usually worthwhile.
**K-consistency**

- Arc consistency does not detect all inconsistencies:
  - Partial assignment \{WA=red, NSW=red\} is inconsistent.

- Stronger forms of propagation can be defined using the notion of k-consistency.

- A CSP is k-consistent if for any set of k-1 variables and for any consistent assignment to those variables, a consistent value can always be assigned to any kth variable.
  - E.g. 1-consistency = node-consistency
  - E.g. 2-consistency = arc-consistency
  - E.g. 3-consistency = path-consistency

- Strongly k-consistent:
  - k-consistent for all values \{k, k-1, ...2, 1\}
Trade-offs

- Running stronger consistency checks...
  - Takes more time
  - But will reduce branching factor and detect more inconsistent partial assignments

  - No “free lunch”
    - In worst case n-consistency takes exponential time

- Generally helpful to enforce 2-Consistency (Arc Consistency)

- Sometimes helpful to enforce 3-Consistency

- Higher levels may take more time to enforce than they save.
Further improvements

• Checking special constraints
  – Checking Alldif(...) constraint
    • *E.g.* \{WA=red, NSW=red\}
  – Checking Atmost(...) constraint
    • *Bounds propagation for larger value domains*

• Intelligent backtracking
  – Standard form is chronological backtracking i.e. try different value for preceding variable.
  – More intelligent, backtrack to conflict set.
    • Set of variables that caused the failure or set of previously assigned variables that are connected to X by constraints.
    • Backjumping moves back to most recent element of the conflict set.
    • Forward checking can be used to determine conflict set.
Local search for CSPs

- Use complete-state representation
  - Initial state = all variables assigned values
  - Successor states = change 1 (or more) values

- For CSPs
  - allow states with unsatisfied constraints (unlike backtracking)
  - operators **reassign** variable values
  - hill-climbing with n-queens is an example

- Variable selection: randomly select any conflicted variable

- Value selection: *min-conflicts heuristic*
  - Select new value that results in a minimum number of conflicts with the other variables
Local search for CSP

function MIN-CONFLICTS(csp, max_steps) return solution or failure

  inputs: csp, a constraint satisfaction problem
           max_steps, the number of steps allowed before giving up

  current ← an initial complete assignment for csp
  for i = 1 to max_steps do
    if current is a solution for csp then return current
    var ← a randomly chosen, conflicted variable from VARIABLES[csp]
    value ← the value v for var that minimize CONFLICTS(var,v,current,csp)
    set var = value in current
  return failure
Min-conflicts example 1

Use of min-conflicts heuristic in hill-climbing.
Min-conflicts example 2

- A two-step solution for an 8-queens problem using min-conflicts heuristic
- At each stage a queen is chosen for reassignment in its column
- The algorithm moves the queen to the min-conflict square breaking ties randomly.
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Zebra: see exercise 6.7 (3rd ed.); exercise 5.13 (2nd ed.)
Advantages of local search

- Local search can be particularly useful in an online setting
  - Airline schedule example
    - E.g., mechanical problems require than 1 plane is taken out of service
    - Can locally search for another “close” solution in state-space
    - Much better (and faster) in practice than finding an entirely new schedule

- The runtime of min-conflicts is roughly independent of problem size.
  - Can solve the millions-queen problem in roughly 50 steps.

  - Why?
    - $n$-queens is easy for local search because of the relatively high density of solutions in state-space
Performance of min-conflicts

Given random initial state, can solve $n$-queens in almost constant time for arbitrary $n$ with high probability (e.g., $n = 10,000,000$)

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$
Hard satisfiability problems
Hard satisfiability problems

- Median runtime for 100 **satisfiable** random 3-CNF sentences, $n = 50$
Sudoku — Backtracking Search + Forward Checking

\[ R = \frac{\text{number of cells}}{\text{number of filled cells}} \]

- Success Rate = \( P(\text{random puzzle is solvable}) \)
- \[ \text{number of cells} = 81 \]
- \[ \text{number of filled cells} = \text{variable} \]

### Avg Time vs. R

![Avg Time vs. R Graph]

### Success Rate vs. R

![Success Rate vs. R Graph]
Graph structure and problem complexity

- Solving disconnected subproblems
  - Suppose each subproblem has $c$ variables out of a total of $n$.
  - Worst case solution cost is $O(n/c^{d^c})$, i.e. linear in $n$
    - Instead of $O(d^n)$, exponential in $n$

- E.g. $n=80$, $c=20$, $d=2$
  - $2^{80} = 4$ billion years at 1 million nodes/sec.
  - $4 \times 2^{20} = .4$ second at 1 million nodes/sec
Tree-structured CSPs

• Theorem:
  – if a constraint graph has no loops then the CSP can be solved in $O(nd^2)$ time
  – linear in the number of variables!

• Compare difference with general CSP, where worst case is $O(d^n)$
Summary

- CSPs
  - special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values

- Backtracking = depth-first search with one variable assigned per node

- Heuristics
  - Variable ordering and value selection heuristics help significantly

- Constraint propagation does additional work to constrain values and detect inconsistencies
  - Works effectively when combined with heuristics

- Iterative min-conflicts is often effective in practice.

- Graph structure of CSPs determines problem complexity
  - e.g., tree structured CSPs can be solved in linear time.