Bayesian Networks

Read R&N Ch. 14.1-14.2

Next lecture: Read R&N 18.1-18.4

You will be expected to know

- Basic concepts and vocabulary of Bayesian networks.
  - Nodes represent random variables.
  - Directed arcs represent (informally) direct influences.
  - Conditional probability tables, P(Xi | Parents(Xi)).

- Given a Bayesian network:
  - Write down the full joint distribution it represents.

- Given a full joint distribution in factored form:
  - Draw the Bayesian network that represents it.

- Given a variable ordering and some background assertions of conditional independence among the variables:
  - Write down the factored form of the full joint distribution, as simplified by the conditional independence assertions.

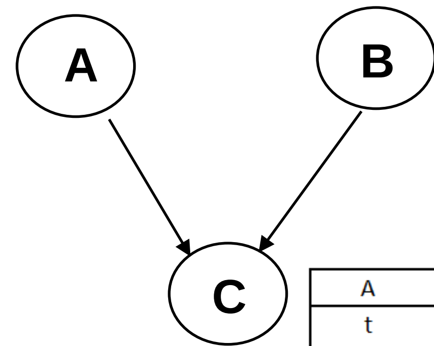
Bayesian Network

- A Bayesian network specifies a joint distribution in a structured form:

$$p(A,B,C) = p(C|A,B)p(A)p(B)$$

P(A)
0.33

P(B)
0.67



A	B	P(C)
t	t	0.2
t	f	0.4
f	t	0.3
f	f	0.3

- Dependence/independence represented via a directed graph:
  - Node = random variable
  - Directed Edge = conditional dependence
  - Absence of Edge = conditional independence

- Allows concise view of joint distribution relationships:
  - Graph nodes and edges show conditional relationships between variables.
  - Tables provide probability data.

Bayesian Networks

- Structure of the graph ⇔ Conditional independence relations

In general,

$$p(X_1, X_2, \ldots X_N) = \prod p(X_i \mid \text{parents}(X_i))$$

The full joint distribution

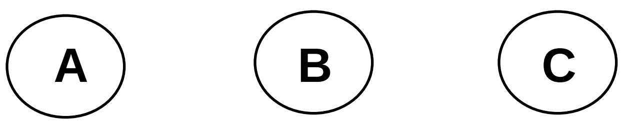
The graph-structured approximation

- Requires that graph is acyclic (no directed cycles)

- 2 components to a Bayesian network
  - The graph structure (conditional independence assumptions)
  - The numerical probabilities (for each variable given its parents)

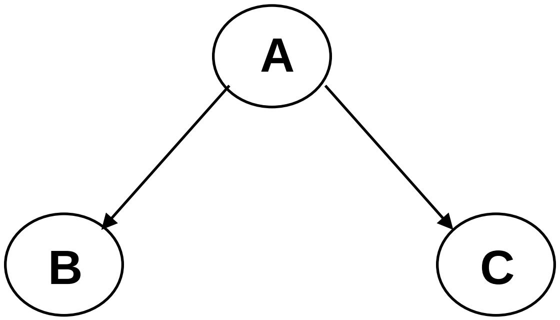
- Also known as belief networks, graphical models, causal networks

Examples of 3-way Bayesian Networks



Marginal Independence:
$p(A,B,C) = p(A)\, p(B)\, p(C)$

Examples of 3-way Bayesian Networks

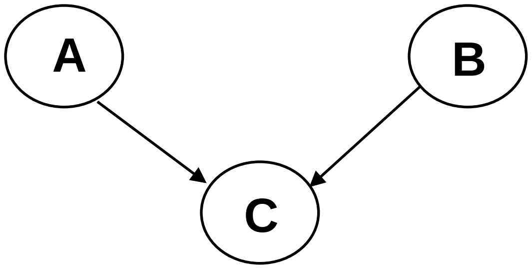


Conditionally independent effects:
$p(A,B,C) = p(B|A)p(C|A)p(A)$

B and C are conditionally independent
Given A

e.g., A is a disease, and we model
B and C as conditionally independent
symptoms given A

Examples of 3-way Bayesian Networks

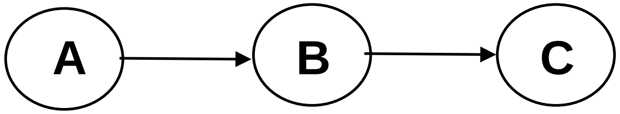


Independent Causes:
p(A,B,C) = p(C|A,B)p(A)p(B)

“Explaining away” effect:
Given C, observing A makes B less likely
e.g., earthquake/burglary/alarm example

A and B are (marginally) independent
but become dependent once C is known

Examples of 3-way Bayesian Networks



Markov dependence:
p(A,B,C) = p(C|B) p(B|A)p(A)

Example

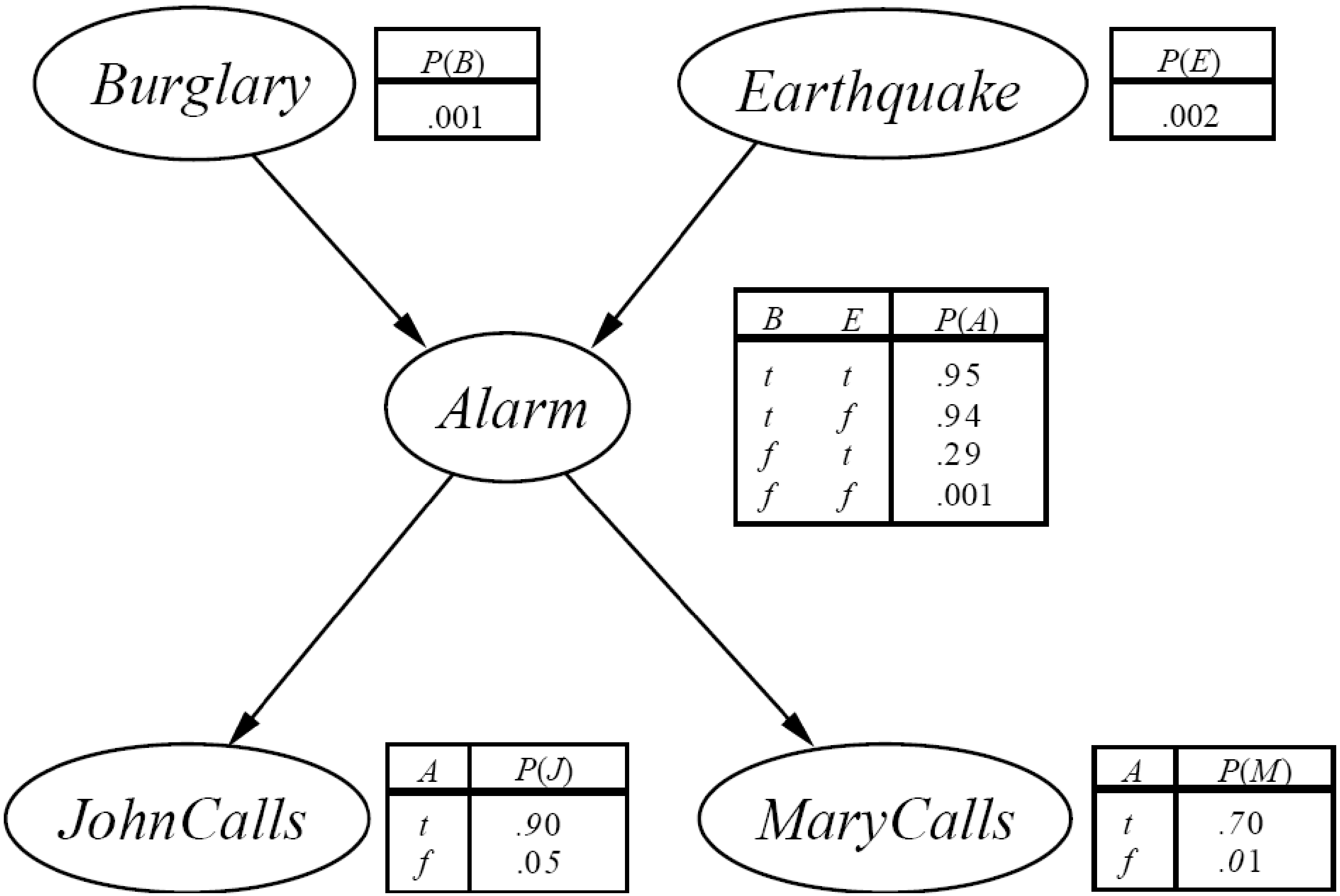
- Consider the following 5 binary variables:
  - B = a burglary occurs at your house
  - E = an earthquake occurs at your house
  - A = the alarm goes off
  - J = John calls to report the alarm
  - M = Mary calls to report the alarm

  - What is P(B | M, J) ? (for example)

  - We can use the full joint distribution to answer this question
    - Requires $2^5 = 32$ probabilities

    - Can we use prior domain knowledge to come up with a Bayesian network that requires fewer probabilities?

The Desired Bayesian Network



$P(B)$
.001

$P(E)$
.002

B	E	$P(A)$
t	t	.95
t	f	.94
f	t	.29
f	f	.001

A	$P(J)$
t	.90
f	.05

A	$P(M)$
t	.70
f	.01

Only requires 10 probabilities!

Constructing a Bayesian Network: Step 1

- Order the variables in terms of influence (may be a partial order)

  e.g., {E, B} -> {A} -> {J, M}

- $P(J, M, A, E, B) = P(J, M \mid A, E, B)\, P(A \mid E, B)\, P(E, B)$

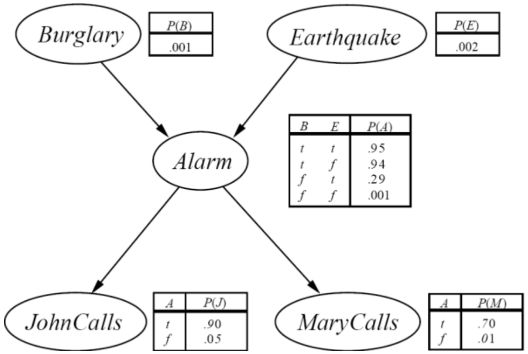
  $\approx P(J, M \mid A) \quad P(A \mid E, B)\, P(E)\, P(B)$

  $\approx P(J \mid A)\, P(M \mid A)\, P(A \mid E, B)\, P(E)\, P(B)$

These conditional independence assumptions are reflected in the graph structure of the Bayesian network

Constructing this Bayesian Network: Step 2

- $P(J, M, A, E, B) =$
  $P(J \mid A)\ P(M \mid A)\ P(A \mid E, B)\ P(E)\ P(B)$

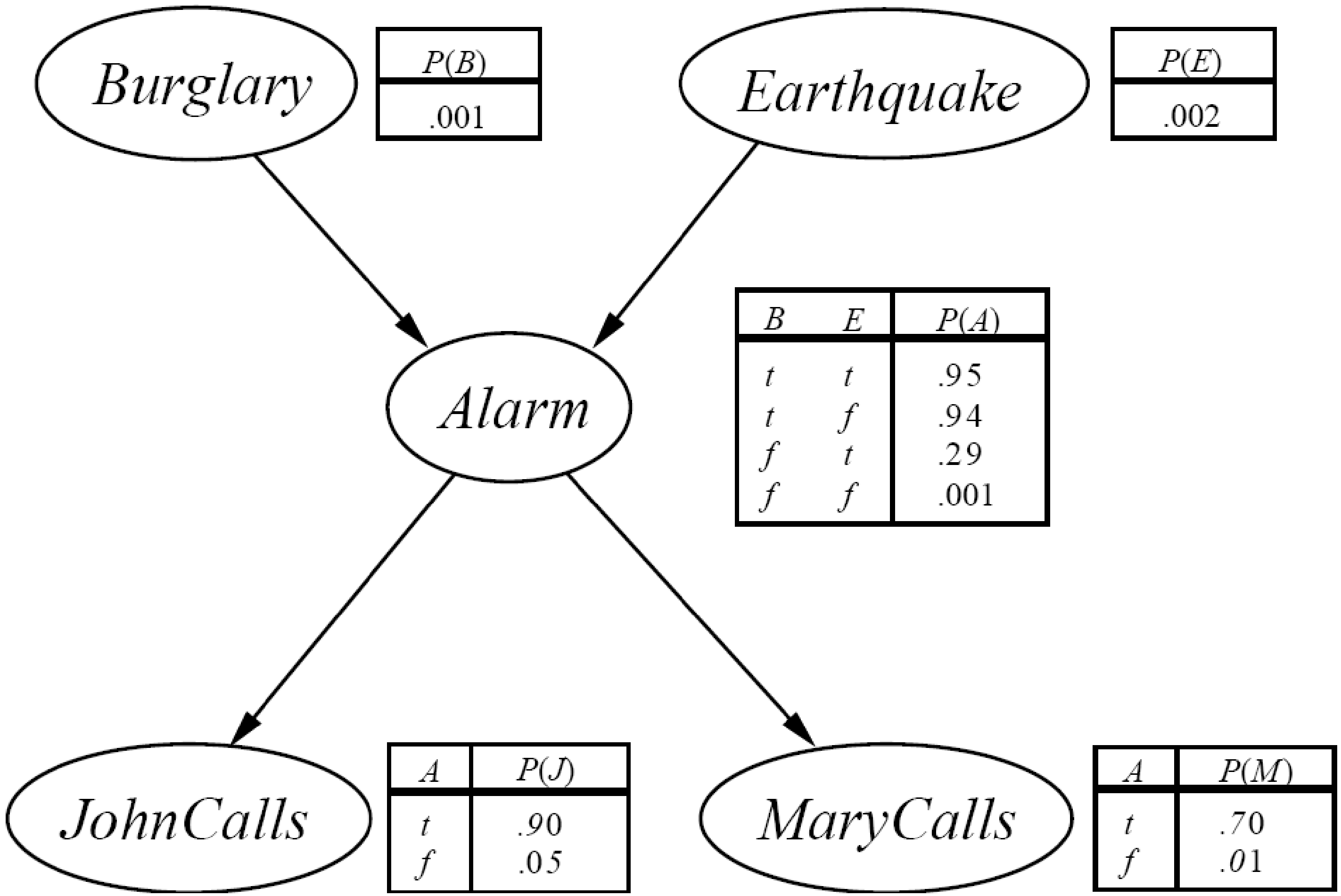


- There are 3 conditional probability tables (CPDs) to be determined: $P(J \mid A)$, $P(M \mid A)$, $P(A \mid E, B)$
  - Requiring 2 + 2 + 4 = 8 probabilities

- And 2 marginal probabilities P(E), P(B) -> 2 more probabilities

- Where do these probabilities come from?
  - Expert knowledge
  - From data (relative frequency estimates)
  - Or a combination of both - see discussion in Section 20.1 and 20.2 (optional)

The Resulting Bayesian Network



$P(B)$
.001

$P(E)$
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B	E	$P(A)$
t	t	.95
t	f	.94
f	t	.29
f	f	.001

A	$P(J)$
t	.90
f	.05

A	$P(M)$
t	.70
f	.01

Example (done the simple, marginalization way)

- So, what is $P(B \mid M, J)$?

  E.g., say, $P(b \mid m, \neg j)$, i.e., $P(B=\text{true} \mid M=\text{true} \wedge J=\text{false})$

$P(b \mid m, \neg j) = P(b, m, \neg j) / P(m, \neg j)$;by definition

$P(b, m, \neg j) = \Sigma A \in \{a, \neg a\} \Sigma E \in \{e, \neg e\}\ P(\neg j, m, A, E, b)$;marginal

$P(J, M, A, E, B) \approx P(J \mid A)\, P(M \mid A)\, P(A \mid E, B)\, P(E)\, P(B)$; conditional indep.

$P(\neg j, m, A, E, b) \approx P(\neg j \mid A)\, P(m \mid A)\, P(A \mid E, b)\, P(E)\, P(b)$

Say, work the case $A=a \wedge E=\neg e$

$P(\neg j, m, a, \neg e, b) \approx P(\neg j \mid a)\, P(m \mid a)\, P(a \mid \neg e, b)\, P(\neg e)\, P(b)$

$\approx 0.10 \times 0.70 \times 0.94 \times 0.998 \times 0.001$

Similar for the cases of $a \wedge e$, $\neg a \wedge e$, $\neg a \wedge \neg e$.

Similar for $P(m, \neg j)$. Then just divide to get $P(b \mid m, \neg j)$.

Number of Probabilities in Bayesian Networks

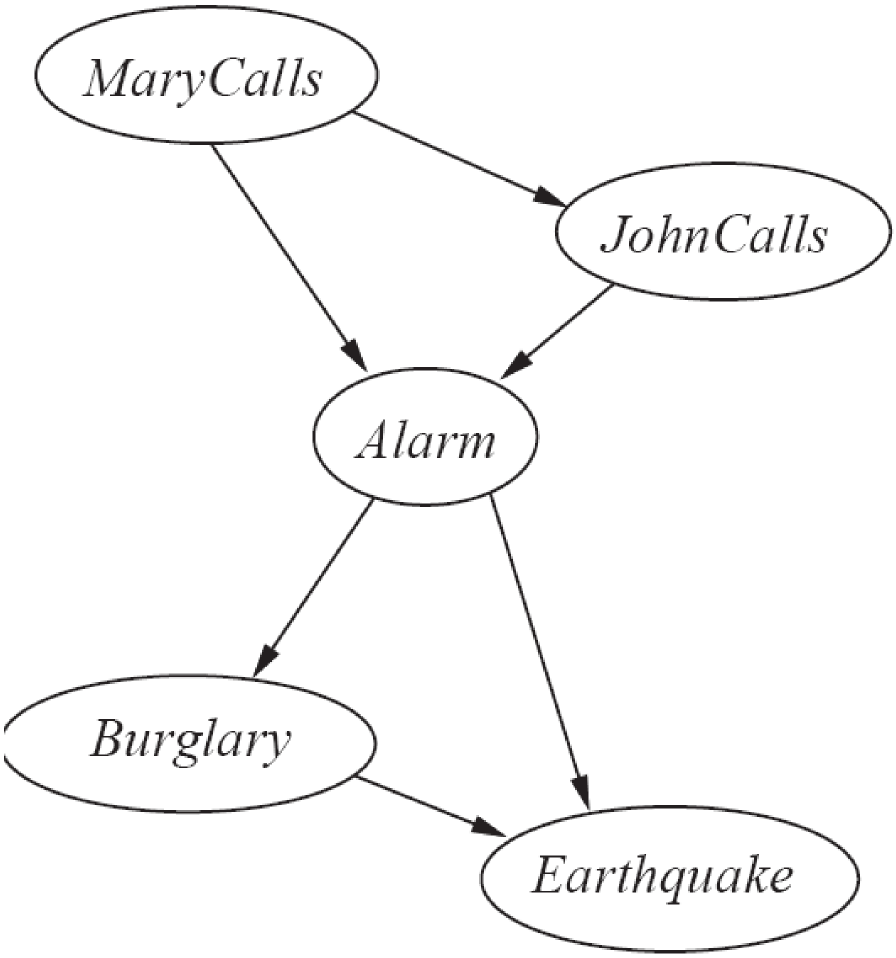
- Consider n binary variables

- Unconstrained joint distribution requires $O(2^n)$ probabilities

- If we have a Bayesian network, with a maximum of k parents for any node, then we need $O(n\ 2^k)$ probabilities

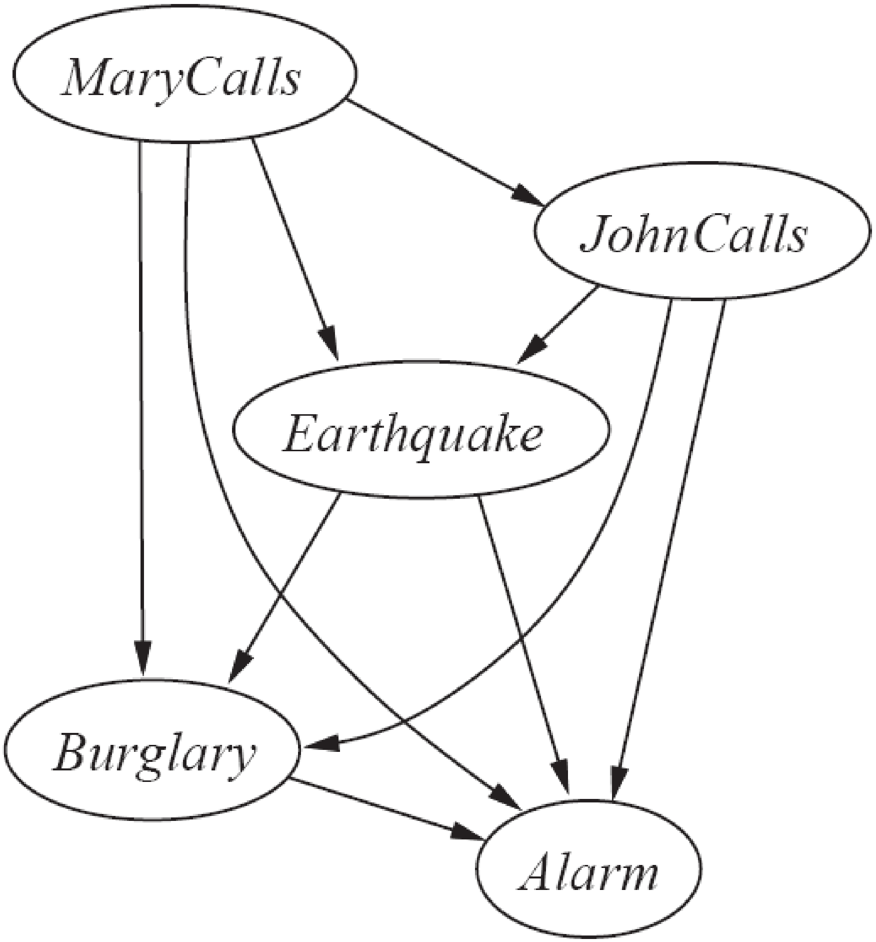
- Example
  - Full unconstrained joint distribution
    - n = 30, k = 4: need 10^9 probabilities for full joint distribution
  - Bayesian network
    - n = 30, k = 4: need 480 probabilities

The Bayesian Network from a different Variable Ordering



(a)

The Bayesian Network from a different Variable Ordering



(b)

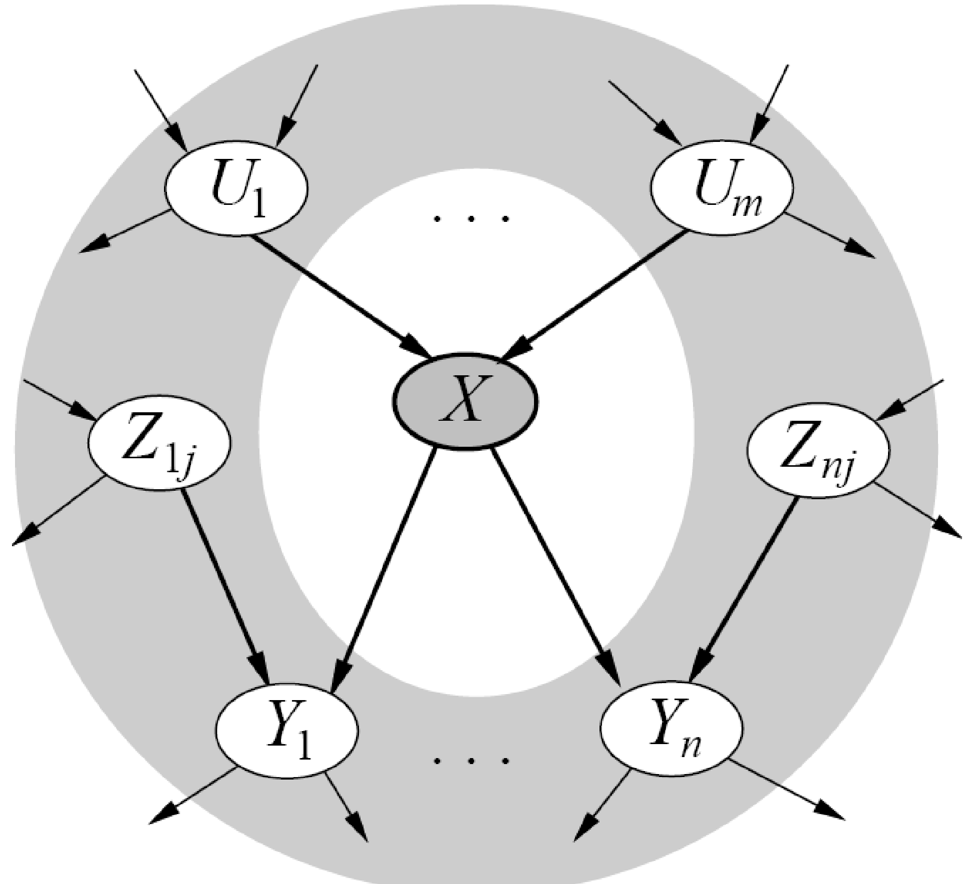
Given a graph, can we “read off” conditional independencies?

The “Markov Blanket” of X (the gray area in the figure)

X is conditionally independent of everything else, GIVEN the values of:

* X’s parents
* X’s children
* X’s children’s parents

X is conditionally independent of its non-descendants, GIVEN the values of its parents.



General Strategy for inference

- Want to compute $P(q \mid e)$

Step 1:

$P(q \mid e) = P(q,e)/P(e) = \alpha P(q,e)$, since $P(e)$ is constant wrt Q

Step 2:

$P(q,e) = \Sigma_{a..z} P(q, e, a, b, \ldots. z)$, by the law of total probability

Step 3:

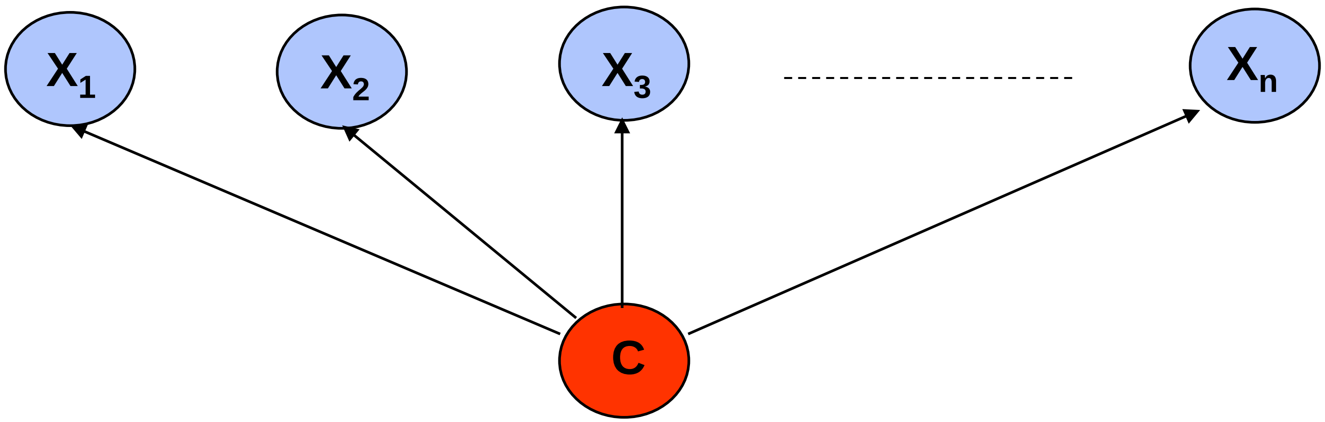
$\Sigma_{a..z} P(q, e, a, b, \ldots. z) = \Sigma_{a..z} \Pi_i P(\text{variable } i \mid \text{parents } i)$

(using Bayesian network factoring)

Step 4:

Distribute summations across product terms for efficient computation

Naïve Bayes Model



$$P(C \mid X_1,\ldots,X_n) = \alpha \prod P(X_i \mid C)\; P(C)$$

Features X are conditionally independent given the class variable C

Widely used in machine learning

e.g., spam email classification: X's = counts of words in emails

Probabilities P(C) and P(Xi | C) can easily be estimated from labeled data

Naïve Bayes Model (2)

$$P(C \mid X_1, \ldots X_n) = \alpha \prod P(X_i \mid C) \; P(C)$$

Probabilities P(C) and P(Xi | C) can easily be estimated from labeled data

P(C = cj) ≈ #(Examples with class label cj) / #(Examples)

P(Xi = xik | C = cj)
≈ #(Examples with Xi value xik and class label cj)
/ #(Examples with class label cj)

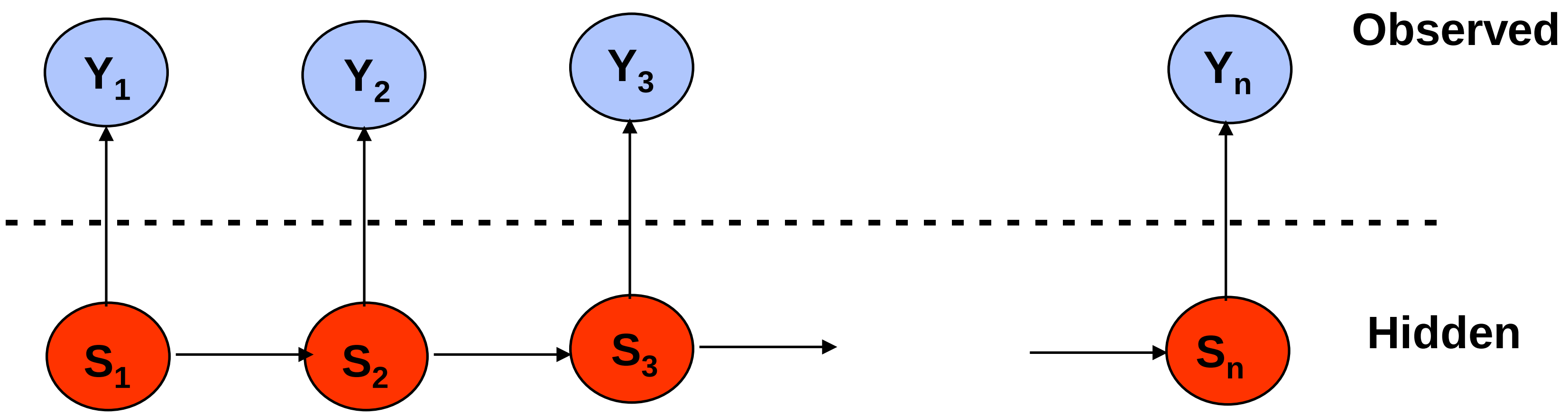
Usually easiest to work with logs

$$\log [\, P(C \mid X_1, \ldots X_n) \,]$$
$$= \log \alpha + \sum [\, \log P(X_i \mid C) + \log P(C) \,]$$

DANGER: Suppose ZERO examples with Xi value xik and class label cj ?
An unseen example with Xi value xik will NEVER predict class label cj !

Practical solutions: Pseudocounts, e.g., add 1 to every #() , etc.
Theoretical solutions: Bayesian inference, beta distribution, etc.

Hidden Markov Model (HMM)



Two key assumptions:

1. hidden state sequence is Markov
2. observation Y_t is CI of all other variables given S_t

Widely used in speech recognition, protein sequence models

Since this is a Bayesian network polytree, inference is linear in n

Summary

- Bayesian networks represent a joint distribution using a graph
- The graph encodes a set of conditional independence assumptions
- Answering queries (or inference or reasoning) in a Bayesian network amounts to efficient computation of appropriate conditional probabilities
- Probabilistic inference is intractable in the general case
  - But can be carried out in linear time for certain classes of Bayesian networks