Informed search algorithms

This lecture topic
Read Chapter 3.5-3.7

Next lecture topic
Read Chapter 4.1-4.2

(Please read lecture topic material before and after each lecture on that topic)
You will be expected to know

- **evaluation function** $f(n)$ and **heuristic function** $h(n)$ for each node $n$
  - $g(n) =$ known path cost so far to node $n$.
  - $h(n) =$ estimate of (optimal) cost to goal from node $n$.
  - $f(n) = g(n) + h(n) =$ estimate of total cost to goal through node $n$.

- Heuristic searches: **Greedy-best-first, A***
  - A* is optimal with admissible (tree)/consistent (graph) heuristics
  - Prove that A* is optimal with admissible heuristic for tree search
  - Recognize when a heuristic is admissible or consistent

- $h_2$ dominates $h_1$ iff $h_2(n) \geq h_1(n)$ for all $n$

- Effective branching factor: $b^*$

- Inventing heuristics: relaxed problems; max or convex combination
Outline

• Review limitations of uninformed search methods

• **Informed (or heuristic) search**
  
  • **Problem-specific heuristics to improve efficiency**
    
    • Best-first, A* (and if needed for memory limits, RBFS, SMA*)
    
    • Techniques for generating heuristics
    
    • A* is optimal with admissible (tree)/consistent (graph) heuristics
    
    • A* is quick and easy to code, and often works *very* well

• **Heuristics**
  
  • A structured way to add “smarts” to your solution
  
  • Provide *significant* speed-ups in practice
  
  • Still have worst-case exponential time complexity

• In AI, “NP-Complete” means “Formally interesting”
Limitations of uninformed search

- Search Space Size makes search tedious
  - Combinatorial Explosion
- For example, 8-puzzle
  - Average solution cost is about 22 steps
  - branching factor ~ 3
  - Exhaustive search to depth 22:
    - $3.1 \times 10^{10}$ states
  - E.g., d=12, IDS expands 3.6 million states on average

[24 puzzle has $10^{24}$ states (much worse)]
Recall tree search...
Recall tree search...

This “strategy” is what differentiates different search algorithms

```python
function TREE-SEARCH(problem, strategy) returns a solution
initialize the search tree using the initial state of problem
loop do
  if there are no candidates for expansion then return failure
  choose a leaf node for expansion according to strategy
  if the node contains a goal state then return the corresponding solution
  else expand the node and add the resulting nodes to the search tree
```

Heuristic search

- **Idea:** use a heuristic function $h(n)$ for each node
  - $g(n) = \text{known path cost so far to node } n$.
  - $h(n) = \text{estimate of (optimal) cost to goal from node } n$.
    - Greedy Best First Search (GBFS) expands the node $n$ with smallest $h(n)$.
  - $f(n) = g(n) + h(n) = \text{estimate of total cost to goal through node } n$.
  - $f(n)$ provides an estimate for the total cost:
    - A* search expands the node $n$ with smallest $f(n)$.

- **Implementation:**
  - Order the nodes in frontier by $h(n)$ for GBFS or by $f(n)$ for A*.

- **Evaluation function is an estimate of node quality**
  - More accurate name for “Greedy best first search” (GBFS) would be “Seemingly best-first search”
  - **Search efficiency depends on heuristic quality!**
    - *The better your heuristic, the faster your search!*
Heuristic function

- **Heuristic:**
  - Definition: a commonsense rule (or set of rules) intended to increase the probability of solving some problem
  - Same linguistic root as “Eureka” = “I have found it”
  - “using rules of thumb to find answers”

- **Heuristic function** $h(n)$
  - Estimate of (optimal) remaining cost from $n$ to goal
  - Defined using only the state of node $n$
  - $h(n) = 0$ if $n$ is a goal node
  - Example: straight line distance from $n$ to Bucharest
    - Note that this is not the true state-space distance
    - It is an estimate – actual state-space distance can be higher

- **Provides problem-specific knowledge to the search algorithm**
Heuristic functions for 8-puzzle

• 8-puzzle
  – Avg. solution cost is about 22 steps
  – branching factor ~ 3
  – Exhaustive search to depth 22:
    • 3.1 x 10^{10} states.
  – A good heuristic function can reduce the search process.

• Two commonly used heuristics
  – \( h_1 \) = the number of misplaced tiles
    • \( h_1(s) = 8 \)
  – \( h_2 \) = the sum of the distances of the tiles from their goal positions (Manhattan distance).
    • \( h_2(s) = 3+1+2+2+2+3+3+2 = 18 \)
Romania with straight-line distance

Values of $h_{SLD}$—straight-line distances to Bucharest.

<table>
<thead>
<tr>
<th>City</th>
<th>Distance</th>
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</thead>
<tbody>
<tr>
<td>Arad</td>
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<td>Vaslui</td>
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<td>Zerind</td>
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</tbody>
</table>
Relationship of Search Algorithms

- $g(n)$ = known cost so far to reach $n$
- $h(n)$ = estimated (optimal) cost from $n$ to goal
- $f(n) = g(n) + h(n)$
  = estimated (optimal) total cost of path through $n$ to goal

- Uniform Cost search sorts frontier by $g(n)$
- Greedy Best First search sorts frontier by $h(n)$
- A* search sorts frontier by $f(n)$
  - Optimal for admissible/consistent heuristics
  - Generally the preferred heuristic search
- Memory-efficient versions of A* are available
  - RBFS, SMA*
Greedy best-first search
(often called just "best-first")

- \( h(n) \) = estimate of cost from \( n \) to goal
  - e.g., \( h(n) \) = straight-line distance from \( n \) to Bucharest

- Greedy best-first search expands the node that appears to be closest to goal.
  - *Priority queue sort function* = \( h(n) \)
Greedy best-first search example
Greedy best-first search example
Greedy best-first search example
Greedy best-first search example
Optimal Path
Greedy Best-first Search
With tree search, will become stuck in this loop

Order of node expansion:  S A D S A D S A D . . . .
Path found:  none  Cost of path found:  none.
Properties of greedy best-first search

- **Complete?**
  - Tree version can get stuck in loops.
  - Graph version is complete in finite spaces.

- **Time?** $O(b^m)$
  - A good heuristic can give **dramatic** improvement

- **Space?** $O(b^m)$
  - Keeps all nodes in memory

- **Optimal?** No
  - E.g., Arad $\rightarrow$ Sibiu $\rightarrow$ Rimnicu Vilcea $\rightarrow$ Pitesti $\rightarrow$ Bucharest is shorter!
**A* search**

- **Idea:** avoid paths that are already expensive
  - Generally the preferred simple heuristic search
  - Optimal if heuristic is:
    - admissible (tree search)/consistent (graph search)

- **Evaluation function** $f(n) = g(n) + h(n)$
  - $g(n) =$ known path cost so far to node $n$.
  - $h(n) =$ estimate of (optimal) cost to goal from node $n$.
  - $f(n) = g(n) + h(n)$
    - $= $ estimate of total cost to goal through node $n$.

- **Priority queue sort function** $= f(n)$
Admissible heuristics

• A heuristic $h(n)$ is admissible if for every node $n$,
  
  $h(n) \leq h^*(n),$

  where $h^*(n)$ is the true cost to reach the goal state from $n$.

• An admissible heuristic never overestimates the cost to reach the goal, i.e., it is optimistic (or at least, never pessimistic)
  – Example: $h_{SLD}(n)$ (never overestimates actual road distance)

• Theorem:
  
  If $h(n)$ is admissible, $A^*$ using $\text{TREE-SEARCH}$ is optimal
Admissible heuristics

E.g., for the 8-puzzle:
- $h_1(n) = \text{number of misplaced tiles}$
- $h_2(n) = \text{total Manhattan distance}$
  (i.e., no. of squares from desired location of each tile)

- $h_1(S) = \ ?$
- $h_2(S) = \ ?$
Admissible heuristics

E.g., for the 8-puzzle:

- $h_1(n) =$ number of misplaced tiles
- $h_2(n) =$ total Manhattan distance
  (i.e., no. of squares from desired location of each tile)

- $h_1(S) =$ ? \(8\)
- $h_2(S) =$ ? \(3+1+2+2+2+3+3+2 = 18\)
Consistent heuristics
(consistent => admissible)

- A heuristic is consistent if for every node \( n \), every successor \( n' \) of \( n \) generated by any action \( a \),
  \[
  h(n) \leq c(n,a,n') + h(n')
  \]

- If \( h \) is consistent, we have
  \[
  f(n') = g(n') + h(n') \quad \text{(by def.)}
  \]
  \[
  = g(n) + c(n,a,n') + h(n') \quad \text{(g(n')=g(n)+c(n,a.n'))}
  \]
  \[
  \geq g(n) + h(n) = f(n) \quad \text{(consistency)}
  \]
  \[
  f(n') \geq f(n)
  \]
  i.e., \( f(n) \) is non-decreasing along any path.

- **Theorem:**
  If \( h(n) \) is consistent, A* using \texttt{GRAPH-SEARCH} is optimal
Admissible (Tree Search) vs. Consistent (Graph Search)

- Why two different conditions?
  - In graph search you often find a long cheap path to a node after a short expensive one, so you might have to update all of its descendants to use the new cheaper path cost so far
  - A consistent heuristic avoids this problem (it can’t happen)
  - Consistent is slightly stronger than admissible
  - Almost all admissible heuristics are also consistent

- Could we do optimal graph search with an admissible heuristic?
  - Yes, but you would have to do additional work to update descendants when a cheaper path to a node is found
  - A consistent heuristic avoids this problem
A* tree search example
A* tree search example:

Simulated queue. City/h/g/f

- Expanded:
- Next:
- Children:
- Frontier: Arad/366/0/366
A* tree search example: Simulated queue. City/h/g/f

Arad/
366/0/366
A* tree search example:
Simulated queue. City/h/g/f

Arad/
366/0/366
A* tree search example:
Simulated queue. City/h/g/f

- Expanded: Arad
- Next: Arad/366/0/366
- Children: Sibiu/253/140/393, Timisoara/329/118/447, Zerind/374/75/449
A* tree search example: Simulated queue. City/h/g/f
A* tree search example:
Simulated queue. City/h/g/f

- Arad/366/0/366
- Sibiu/253/140/393
- Timisoara/329/118/447
- Zerind/374/75/449
A* tree search example
A* tree search example: Simulated queue. City/h/g/f

- **Expanded:** Arad, Sibiu
- **Next:** Sibiu/253/140/393
- **Children:** Arad/366/280/646, Fagaras/176/239/415, Oradea/380/291/671, RimnicuVilcea/193/220/413
A* tree search example: Simulated queue. City/h/g/f
A* tree search example:
Simulated queue. City/h/g/f
A* tree search example
A* tree search example:

Simulated queue. City/h/g/f

- **Expanded:** Arad, Sibiu, RimnicuVilcea
- **Next:** RimnicuVilcea/193/220/413
- **Children:** Craiova/160/368/528, Pitesti/100/317/417, Sibiu/253/300/553
A* tree search example: Simulated queue. City/h/g/f

- **Arad**: 366/0/366
- **Sibiu**: 253/140/393
  - **Arad**: 366/280/646
  - **Fagaras**: 176/239/415
  - **Oradea**: 380/291/671
  - **Craiova**: 160/368/528
- **Timisoara**: 329/118/447
- **Zerind**: 374/75/449
  - **Rimnicu Vilcea**: 193/220/413
  - **Pitesti**: 100/317/417
  - **Sibiu**: 253/300/553
A* search example:
Simulated queue. City/h/g/f
A* tree search example
A* tree search example:
Simulated queue. City/h/g/f

- **Expanded**: Arad, Sibiu, RimnicuVilcea, Fagaras
- **Next**: Fagaras/176/239/415
- **Children**: Bucharest/0/579/579, Sibiu/253/338/591
A* tree search example
A* tree search example:
Simulated queue. City/h/g/f

- **Expanded**: Arad, Sibiu, RimnicuVilcea, Fagaras, Pitesti
- **Next**: Pitesti/100/317/417
- **Children**: Bucharest/0/418/418, Craiova/160/455/615, RimnicuVilcea/193/414/607
A* tree search example
A* tree search example:
Simulated queue. City/h/g/f

- **Expanded:** Arad, Sibiu, RimnicuVilcea, Fagaras, Pitesti, Bucharest
- **Next:** Bucharest/0/418/418
- **Children:** None; goal test succeeds.
A* tree search example:
Simulated queue. City/h/g/f
A* tree search example:
Simulated queue. City/h/g/f
Contours of A* Search

- A* expands nodes in order of increasing $f$ value
- Gradually adds "$f$-contours" of nodes
- Contour $i$ has all nodes with $f=f_i$, where $f_i < f_{i+1}$
Properties of A*

• **Complete?** Yes
  (unless there are infinitely many nodes with \( f \leq f(G) \); can’t happen if step-cost \( \geq \varepsilon > 0 \))

• **Time/Space?** Exponential \( O(b^d) \)
  except if:
  \[
  | h(n) - h^*(n) | \leq O(\log h^*(n))
  \]

• **Optimal?** Yes
  (with: Tree-Search, admissible heuristic;
  Graph-Search, consistent heuristic)

• **Optimally Efficient?** Yes
  (no optimal algorithm with same heuristic is guaranteed to expand fewer nodes)
Optimality of A* (proof)
Tree Search, where $h(n)$ is admissible

- Suppose some suboptimal goal $G_2$ has been generated and is in the frontier. Let $n$ be an unexpanded node in the frontier such that $n$ is on a shortest path to an optimal goal $G$.

\[
\begin{align*}
\text{We want to prove:} & \quad f(n) < f(G_2) \\
& \quad \text{(then A* will expand n before G2)}
\end{align*}
\]

- $f(G_2) = g(G_2)$ since $h(G_2) = 0$
- $f(G) = g(G)$ since $h(G) = 0$
- $g(G_2) > g(G)$ since $G_2$ is suboptimal
- $f(G_2) > f(G)$ from above, with $h=0$
- $h(n) \leq h^*(n)$ since $h$ is admissible (under-estimate)
- $g(n) + h(n) \leq g(n) + h^*(n)$ from above
- $f(n) \leq f(G)$ since $g(n)+h(n)=f(n) \ & \ g(n)+h^*(n)=f(G)$
- $f(n) < f(G_2)$ from above
Memory Bounded Heuristic Search: Recursive Best First Search (RBFS)

• How can we solve the memory problem for A* search?

• **Idea**: Try something like depth first search, but let’s not forget everything about the branches we have partially explored.

• **We remember the best f(n) value we have found so far in the branch we are deleting.**
RBFS changes its mind very often in practice.

This is because the $f=g+h$ become more accurate (less optimistic) as we approach the goal. Hence, higher level nodes have smaller f-values and will be explored first.

Problem: We should keep in memory whatever we can.
Simple Memory Bounded A* (SMA*)

- This is like A*, but when memory is full we delete the worst node (largest f-value).
- Like RBFS, we remember the best descendent in the branch we delete.
- If there is a tie (equal f-values) we delete the oldest nodes first.
- simple-MBA* finds the optimal reachable solution given the memory constraint.
- Time can still be exponential.

A Solution is not reachable if a single path from root to goal does not fit into memory.
**SMA* pseudocode (not in 2nd edition of R&N)**

```
function SMA*(problem) returns a solution sequence
inputs: problem, a problem
static: Queue, a queue of nodes ordered by f-cost

Queue ← MAKE-QUEUE({MAKE-NODE(INITIAL-STATE[problem])})
loop do
    if Queue is empty then return failure
    n ← deepest least-f-cost node in Queue
    if GOAL-TEST(n) then return success
    s ← NEXT-SUCCESSOR(n)
    if s is not a goal and is at maximum depth then
        f(s) ← ∞
    else
        f(s) ← MAX(f(n), g(s) + h(s))
    if all of n’s successors have been generated then
        update n’s f-cost and those of its ancestors if necessary
    if SUCCESSORS(n) all in memory then remove n from Queue
    if memory is full then
        delete shallowest, highest-f-cost node in Queue
        remove it from its parent’s successor list
        insert its parent on Queue if necessary
        insert s in Queue
end
```
Simple Memory-bounded A* (SMA*)

(Example with 3-node memory)

Progress of SMA*. Each node is labeled with its current $f$-cost. Values in parentheses show the value of the best forgotten descendant.

Search space

$g + h = f$

$\square$ = goal

Algorithm can tell you when best solution found within memory constraint is optimal or not.
Memory Bounded A* Search

- The Memory Bounded A* Search is the best of the search algorithms we have seen so far. It uses all its memory to avoid double work and uses smart heuristics to first descend into promising branches of the search-tree.

- If memory not a problem, then plain A* search is easy to code and performs well.
Heuristic functions

- 8-puzzle
  - Avg. solution cost is about 22 steps
  - branching factor ~ 3
  - Exhaustive search to depth 22:
    - $3.1 \times 10^{10}$ states.
  - A good heuristic function can reduce the search process.

- Two commonly used heuristics
  - $h_1$ = the number of misplaced tiles
    - $h_1(s) = 8$
  - $h_2$ = the sum of the axis-parallel distances of the tiles from their goal positions (Manhattan distance).
    - $h_2(s) = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18$
Dominance

• IF $h_2(n) \geq h_1(n)$ for all $n$
  THEN $h_2$ dominates $h_1$
  – $h_2$ is almost always better for search than $h_1$
  – $h_2$ guarantees to expand no more nodes than does $h_1$
  – $h_2$ almost always expands fewer nodes than does $h_1$
  – Not useful unless both $h_1$ & $h_2$ are admissible/consistent

• Typical 8-puzzle search costs
  (average number of nodes expanded):
  – $d=12$ $\text{IDS} = 3,644,035$ nodes
    $A^*(h_1) = 227$ nodes
    $A^*(h_2) = 73$ nodes
  – $d=24$ $\text{IDS} = \text{too many nodes}$
    $A^*(h_1) = 39,135$ nodes
    $A^*(h_2) = 1,641$ nodes
Idea: Heuristic for “Go to Bucharest” that dominates SLD

• Basic idea:
  – \( h(n) \) {
    IF (you can get to Bucharest from \( n \) in one step), THEN (you’re done)
    ELSE /* you have to go through at least one other city */
       RETURN ( MIN\_all\_cities \( C \), hSLD(n,C)+hSLD(C,Bucharest) );
  }

• Embellishments (Hierarchy)
  – \( h(n, k) \) {
    IF (you can get to Bucharest (B) from \( n \) in \( k \) or fewer steps),
       THEN RETURN \( h(n, k-1) \)
    ELSE /* you have to go through at least \( k \) cities */
       RETURN( MIN\_all\_k\_cities\_combos \( C \), hSLD(n through C to B) );
  }

• In practice, you also have to check that your path of length \( k \) is not a short expensive path (see previous)
  – In practice, you must take the MIN over all \( j \leq k \) cities.
Heuristic for “Go to Bucharest” that dominates SLD

- Array $A[i,j] =$ straight-line distance (SLD) from city $i$ to city $j$; $B =$ Bucharest;
- $s(n) =$ successors of $n$;
- $c(m,n) =$ \{if ($n$ in $s(m)$) then (one-step road distance m to n) else $+\infty$\};
- $s_k(n) =$ all descendants of $n$ accessible from $n$ in exactly $k$ steps;
- $S_k(n) =$ all descendants of $n$ accessible from $n$ in $k$ steps or less;
- $C_k(m,n)$
  = \{if ($n$ in $S_k(m)$) then (shortest road distance m to n in $k$ steps or less) else $+\infty$\};
- $s$, $c$, are computable in $O(b)$; $s_k$, $S_k$, $C_k$, are computable in $O(b^k)$.

- These heuristics both dominate SLD, and $h_2$ dominates $h_1$:
  - $h_1(n) =$ $\min_{x \text{ in Romania}} \ (A[n,x] + A[x,B])$
  - $h_2(n) =$ $\min_{x \text{ in } s(n)} \ (c(n,x) + A[x,B])$

- This family of heuristics all dominate SLD, and $i>j \Rightarrow h_i$ dominates $h_j$:
  - $h_k(n) =$ $\min( \min_{x \text{ in } (S_k(n) \cap S_k(B))} \ C_k(n,x)+C_k(x,B)),
  \ (min_{x \text{ in } s_k(n), y \text{ in } s_k(B)} \ (C_k(n,x) + A[x,y] + C_k(y,B)))$
  - $h_{\text{final}}(n)$ = same as bidirectional search; $\Rightarrow$ exponential cost
Effective branching factor: $b^*$

- Let $A^*$ generate $N$ nodes to find a goal at depth $d$
  - $b^*$ is the branching factor that a uniform tree of depth $d$ would have in order to contain $N+1$ nodes.

\[
N + 1 = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d
\]
\[
N + 1 = ((b^*)^{d+1} - 1) / (b^* - 1)
\]
\[
N \approx (b^*)^d \Rightarrow b^* \approx \sqrt[d]{N}
\]

- For sufficiently hard problems, the measure $b^*$ usually is fairly constant across different problem instances.

- A good guide to the heuristic’s overall usefulness.
- A good way to compare different heuristics.
Effective Branching Factor
Pseudo-code (Binary search)

- PROCEDURE EFFBRANCH (START, END, N, D, DELTA)
  COMMENT DELTA IS A SMALL POSITIVE NUMBER FOR ACCURACY OF RESULT.
  MID := (START + END) / 2.
  IF (END - START < DELTA)
    THEN RETURN (MID).
  TEST := EFFPOLY (MID, D).
  IF (TEST < N+1)
    THEN RETURN (EFFBRANCH (MID, END, N, D, DELTA) )
  ELSE RETURN (EFFBRANCH (START, MID, N, D, DELTA) ).
END EFFBRANCH.

PROCEDURE EFFPOLY (B, D)
  ANSWER = 1.
  TEMP = 1.
  FOR I FROM 1 TO (D-1) DO
    TEMP := TEMP * B.
    ANSWER := ANSWER + TEMP.
  ENDDO.
  RETURN (ANSWER).
END EFFPOLY.

- For binary search please see: http://en.wikipedia.org/wiki/Binary_search_algorithm
- An attractive alternative is to use Newton's Method (next lecture) to solve for the root (i.e., f(b)=0) of
  \[ f(b) = 1 + b + \ldots + b^d - (N+1) \]
Effectiveness of different heuristics

- Results averaged over random instances of the 8-puzzle

```
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<th>A*(h2)</th>
<th>IDS</th>
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Inventing heuristics via “relaxed problems”

- A problem with fewer restrictions on the actions is called a relaxed problem.

- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem.

- If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution.

- If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution.

- Can be a useful way to generate heuristics.
  - E.g., ABSOLVER (Prieditis, 1993) discovered the first useful heuristic for the Rubik’s cube puzzle.
More on heuristics

- \( h(n) = \max\{ h_1(n), h_2(n), \ldots, h_k(n) \} \)
  - Assume all \( h \) functions are admissible
  - E.g., \( h_1(n) = \# \) of misplaced tiles
  - E.g., \( h_2(n) = \) manhattan distance, etc.
  - \( \max \) chooses least optimistic heuristic (most accurate) at each node

- \( h(n) = w_1 h_1(n) + w_2 h_2(n) + \ldots + w_k h_k(n) \)
  - A convex combination of features
    - Weighted sum of \( h(n) \)'s, where weights sum to 1
  - Weights learned via repeated puzzle-solving
  - Try to identify which features are predictive of path cost
Admissible heuristics can also be derived from the solution cost of a subproblem of a given problem. This cost is a lower bound on the cost of the real problem. Pattern databases store the exact solution to for every possible subproblem instance. The complete heuristic is constructed using the patterns in the DB.
An Admissible but Inconsistent Heuristic For the 8-puzzle (interesting side note)

• \( h1 \) = Pattern Database for tiles 1,2,3,4
  – Obviously, \( h1 \) is both admissible & consistent

• \( h2 \) = Pattern Database for tiles 5,6,7,8
  – Obviously, \( h2 \) is both admissible & consistent

• \( h(n) = \text{choose\_randomly}(\ h1(n), \ h2(n) \ ) \)
  – \( h \) is admissible but not (necessarily) consistent
  – \( h \) is (probably) not non-decreasing along all paths
    • \( h1 \) and \( h2 \) are not necessarily related to each other
    • Random combination may not satisfy triangle inequality

Example adapted from “Inconsistent Heuristics in Theory and Practice” by Felner, Zahavi, Holte, Schaeffer, Sturtevant, & Zhang
Summary

• Uninformed search methods have uses, also severe limitations
• Heuristics are a structured way to add “smarts” to your search

• Informed (or heuristic) search uses problem-specific heuristics to improve efficiency
  • Best-first, A* (and if needed for memory limits, RBFS, SMA*)
  • Techniques for generating heuristics
  • A* is optimal with admissible (tree)/consistent (graph) heuristics

• Can provide significant speed-ups in practice
  • E.g., on 8-puzzle, speed-up is dramatic
  • Still have worst-case exponential time complexity
  • In AI, “NP-Complete” means “Formally interesting”

• Next lecture topic: local search techniques
  • Hill-climbing, genetic algorithms, simulated annealing, etc.
  • Read Chapter 4 in advance of lecture, and again after lecture