Inference in First-Order Logic
Outline

- Reducing first-order inference to propositional inference
- Unification
- Generalized Modus Ponens
- Forward chaining
- Backward chaining
- Resolution
- Other types of reasoning
  - Induction, abduction, analogy
  - Modal logics
You will be expected to know

- Basic concepts and vocabulary of unification.

- Given two terms containing variables
  - Find the most general unifier if one exists.
  - Else, explain why no unification is possible.
  - See figure 9.1 and surrounding text in your textbook.
Universal instantiation (UI)

- Notation: \( \text{Subst}\{\{v/g\}, \alpha\} \) means the result of substituting ground term \( g \) for variable \( v \) in sentence \( \alpha \)

- Every instantiation of a universally quantified sentence is entailed by it:

\[
\frac{\forall v \alpha}{\text{Subst}\{\{v/g\}, \alpha\}}
\]

for any variable \( v \) and ground term \( g \)

- E.g., \( \forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x) \) yields:

\[
\begin{align*}
\text{King}(John) \land \text{Greedy}(John) & \Rightarrow \text{Evil}(John), \quad \{x/John\} \\
\text{King}(Richard) \land \text{Greedy}(Richard) & \Rightarrow \text{Evil}(Richard), \quad \{x/Richard\} \\
\text{King}(\text{Father}(John)) \land \text{Greedy}(\text{Father}(John)) & \Rightarrow \text{Evil}(\text{Father}(John)), \quad \{x/\text{Father}(John)\}
\end{align*}
\]
Existential instantiation (EI)

- For any sentence $\alpha$, variable $v$, and constant symbol $k$ (that does not appear elsewhere in the knowledge base):

$$\exists v \alpha \quad \text{Subst}(\{v/k\}, \alpha)$$

- E.g., $\exists x\ Crown(x) \land OnHead(x, John)$ yields:

$$Crown(C_1) \land OnHead(C_1, John)$$

where $C_1$ is a new constant symbol, called a Skolem constant

- Existential and universal instantiation allows to “propositionalize” any FOL sentence or KB
  - EI produces one instantiation per EQ sentence
  - UI produces a whole set of instantiated sentences per UQ sentence
Reduction to propositional form

Suppose the KB contains the following:

$$\forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x)$$
King(John)
Greedy(John)
Brother(Richard,John)

- Instantiating the universal sentence in all possible ways, we have:
  (there are only two ground terms: John and Richard)

  King(John) \land \text{ Greedy}(John) \Rightarrow \text{ Evil}(John)
  King(Richard) \land \text{ Greedy}(Richard) \Rightarrow \text{ Evil}(Richard)
  King(John)
  Greedy(John)
  Brother(Richard,John)

- The new KB is **propositionalized** with “propositions”:

  King(John), \text{ Greedy}(John), \text{ Evil}(John), \text{ King}(Richard), \text{ etc.}
Reduction continued

- Every FOL KB can be propositionalized so as to preserve entailment
  - A ground sentence is entailed by new KB iff entailed by original KB

- Idea for doing inference in FOL:
  - propositionalize KB and query
  - apply resolution-based inference
  - return result

- Problem: with function symbols, there are infinitely many ground terms,
  - e.g., \textit{Father(Father(Father(John)))}, etc
Reduction continued

Theorem: Herbrand (1930). If a sentence $\alpha$ is entailed by a FOL KB, it is entailed by a finite subset of the propositionalized KB

Idea: For $n = 0$ to $\infty$ do
create a propositional KB by instantiating with depth $n$ terms
see if $\alpha$ is entailed by this KB

Problem: works if $\alpha$ is entailed, loops if $\alpha$ is not entailed.
$\rightarrow$ The problem of **semi-decidable**: algorithms exist
to prove entailment, but no algorithm exists
to prove non-entailment for every non-entailed sentence.
Other Problems with Propositionalization

- Propositionalization generates lots of irrelevant sentences
  - So inference may be very inefficient

- e.g., from:

$$\forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x)$$

$$\text{ King}(\text{John})$$

$$\forall y \text{ Greedy}(y)$$

$$\text{ Brother}(\text{Richard}, \text{John})$$

- it seems obvious that $\text{ Evil}(\text{John})$ is entailed, but propositionalization produces lots of facts such as $\text{ Greedy}(\text{Richard})$ that are irrelevant

- With $p$ $k$-ary predicates and $n$ constants, there are $p \cdot n^k$ instantiations

- Lets see if we can do inference directly with FOL sentences
Unification

• Recall: Subst(θ, p) = result of substituting θ into sentence p

• Unify algorithm: takes 2 sentences p and q and returns a unifier if one exists

  Unify(p,q) = θ  where Subst(θ, p) = Subst(θ, q)

• Example:
  p = Knows(John,x)
  q = Knows(John, Jane)

  Unify(p,q) = {x/Jane}
### Unification examples

- **simple example:** query = `Knows(John,x)`, i.e., who does John know?

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>θ</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>Knows(John,x)</code></td>
<td><code>Knows(John,Jane)</code></td>
<td><code>{x/Jane}</code></td>
</tr>
<tr>
<td><code>Knows(John,x)</code></td>
<td><code>Knows(y,OJ)</code></td>
<td><code>{x/OJ,y/John}</code></td>
</tr>
<tr>
<td><code>Knows(John,x)</code></td>
<td><code>Knows(y,Mother(y))</code></td>
<td><code>{y/John,x/Mother(John)}</code></td>
</tr>
<tr>
<td><code>Knows(John,x)</code></td>
<td><code>Knows(x,OJ)</code></td>
<td><code>{fail}</code></td>
</tr>
</tbody>
</table>

- Last unification fails: only because x can’t take values John and OJ at the same time
  - But we know that if John knows x, and everyone (x) knows OJ, we should be able to infer that John knows OJ

- Problem is due to use of same variable x in both sentences

- Simple solution: Standardizing apart eliminates overlap of variables, e.g., `Knows(z,OJ)`
Unification

• To unify $\text{Knows}(John,x)$ and $\text{Knows}(y,z)$,

$$\theta = \{y/\text{John}, \ x/z \} \text{ or } \theta = \{y/\text{John}, \ x/\text{John}, \ z/\text{John}\}$$

• The first unifier is more general than the second.

• There is a single most general unifier (MGU) that is unique up to renaming of variables.

$$\text{MGU} = \{y/\text{John},\ x/z\}$$

• General algorithm in Figure 9.1 in the text
Recall our example...

\( \forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x) \)

King(John)

\( \forall y \ Greedy(y) \)

Brother(Richard,John)

And we would like to infer Evil(John) without propositionalization
Generalized Modus Ponens (GMP)

\[ p_1', p_2', \ldots, p_n', (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q) \]

\[ \text{Subst}(\theta, q) \]

where we can unify \( p_i' \) and \( p_i \) for all \( i \)

Example:

\( p_1' \) is \( \text{King}(John) \)  \( p_1 \) is \( \text{King}(x) \)
\( p_2' \) is \( \text{Greedy}(y) \)  \( p_2 \) is \( \text{Greedy}(x) \)
\( \theta \) is \{x/John,y/John\}  \( q \) is \( \text{Evil}(x) \)
Subst(\( \theta \),\( q \)) is \( \text{Evil}(John) \)

- Implicit assumption that all variables universally quantified
Completeness and Soundness of GMP

• GMP is sound
  – Only derives sentences that are logically entailed

• GMP is complete for a KB consisting of definite clauses
  – Complete: derives all sentences that are entailed
  – OR...answers every query whose answers are entailed by such a KB

  – Definite clause: disjunction of literals of which exactly 1 is positive,
    e.g., King(x) AND Greedy(x) -> Evil(x)
    NOT(King(x)) OR NOT(Greedy(x)) OR Evil(x)
Inference approaches in FOL

- **Forward-chaining**
  - Uses GMP to add new atomic sentences
  - Useful for systems that make inferences as information streams in
  - Requires KB to be in form of first-order definite clauses

- **Backward-chaining**
  - Works backwards from a query to try to construct a proof
  - Can suffer from repeated states and incompleteness
  - Useful for query-driven inference

- **Resolution-based inference (FOL)**
  - Refutation-complete for general KB
    - Can be used to confirm or refute a sentence p (but not to generate all entailed sentences)
  - Requires FOL KB to be reduced to CNF
  - Uses generalized version of propositional inference rule

- Note that all of these methods are generalizations of their propositional equivalents
Knowledge Base in FOL

- The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

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... it is a crime for an American to sell weapons to hostile nations:

\[\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x,y,z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)\]

Nono ... has some missiles, i.e., \(\exists x \text{ Owns}(\text{Nono},x) \land \text{Missile}(x)\):

\(\text{Owns}(\text{Nono},M_{1}) \text{ and } \text{Missile}(M_{1})\)

... all of its missiles were sold to it by Colonel West

\(\text{Missile}(x) \land \text{Owns}(\text{Nono},x) \Rightarrow \text{Sells}(\text{West},x,\text{Nono})\)

Missiles are weapons:

\(\text{Missile}(x) \Rightarrow \text{Weapon}(x)\)

An enemy of America counts as "hostile":

\(\text{Enemy}(x,\text{America}) \Rightarrow \text{Hostile}(x)\)

West, who is American ...

\(\text{American}(\text{West})\)

The country Nono, an enemy of America ...

\(\text{Enemy}(\text{Nono},\text{America})\)
Forward chaining proof

American(West)  Missile(M1)  Owns(Nono,M1)  Enemy(Nono,America)
Forward chaining proof
Forward chaining proof

```
Criminal(West)

Weapon(M1)  Sells(West,M1,Nono)  Hostile(Nono)

American(West)  Missile(M1)  Owns(Nono,M1)  Enemy(Nono,America)
```
Properties of forward chaining

- Sound and complete for first-order definite clauses
- Datalog = first-order definite clauses + no functions
- FC terminates for Datalog in finite number of iterations
- May not terminate in general if α is not entailed
- Incremental forward chaining: no need to match a rule on iteration $k$ if a premise wasn't added on iteration $k-1$
  ⇒ match each rule whose premise contains a newly added positive literal
Hard matching example

To unify the grounded propositions with premises of the implication you need to solve a CSP!

- Colorable() is inferred iff the CSP has a solution
- CSPs include 3SAT as a special case, hence matching is NP-hard

\[
\begin{align*}
\text{Diff}(\text{wa,nt}) & \land \text{Diff}(\text{wa,sa}) & \land \text{Diff}(\text{nt,q}) & \land \\
\text{Diff}(\text{nt,sa}) & \land \text{Diff}(\text{q,nsw}) & \land \text{Diff}(\text{q,sa}) & \land \\
\text{Diff}(\text{nsw,v}) & \land \text{Diff}(\text{nsw,sa}) & \land \text{Diff}(\text{v,sa}) & \Rightarrow \\
\text{Colorable}() \\
\text{Diff}(\text{Red,Blue}) & \land \text{Diff}(\text{Red,Green}) \\
\text{Diff}(\text{Green,Red}) & \land \text{Diff}(\text{Green,Blue}) \\
\text{Diff}(\text{Blue,Red}) & \land \text{Diff}(\text{Blue,Green})
\end{align*}
\]
Backward chaining example

Criminal(West)
Backward chaining example

```
Criminal(West)
  /   \     \{x/West\}
 /     \     /
American(x) Weapon(y) Sells(x,y,z)
  \       \     /
       \   \   
         \ Hostile(z)
```
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example
Backward chaining example

```
Criminal(West)

American(West) { }
Weapon(y) { }
Sells(West, M1, z) { z/Nono }

Missile(y) { y/M1 }
Missile(M1) { }
Owns(Nono, M1) { }

Hostile(Nono) { }
Enemy(Nono, America) { }
```
Properties of backward chaining

- Depth-first recursive proof search:
  - Space is linear in size of proof.

- Incomplete due to infinite loops
  - ⇒ fix by checking current goal against every goal on stack

- Inefficient due to repeated subgoals (both success and failure)
  - ⇒ fix using caching of previous results (memoization)

- Widely used for logic programming

- PROLOG:
  backward chaining with Horn clauses + bells & whistles.
Resolution in FOL

• Full first-order version:

\[
\begin{array}{c}
\ell_1 \lor \cdots \lor \ell_k, \\
\cdots \\
\ell_i \lor \cdots \\
\cdots \\
m_1 \lor \cdots \lor m_n
\end{array}
\]

Subst(\(\theta\), \(\ell_i \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n\))

where \(\text{Unify}(\ell_i, \neg m_j) = \theta\).

• The two clauses are assumed to be standardized apart so that they share no variables.

• For example,

\[
\begin{array}{c}
\neg \text{Rich}(x) \lor \text{Unhappy}(x), \\
\text{Rich}(\text{Ken})
\end{array}
\]

\[
\text{Unhappy}(\text{Ken})
\]

with \(\theta = \{x/\text{Ken}\}\)

• Apply resolution steps to \(\text{CNF}(\KB \land \neg a)\); complete for FOL
Converting FOL sentences to CNF

Original sentence:
Everyone who loves all animals is loved by someone:
\[ \forall x \left( \forall y \ Animal(y) \Rightarrow Loves(x,y) \right) \Rightarrow \exists y \ Loves(y,x) \]

1. Eliminate biconditionals and implications

\[ \forall x \left( \neg \forall y \ \neg Animal(y) \lor Loves(x,y) \right) \lor \exists y \ Loves(y,x) \]

2. Move \( \neg \) inwards:
Recall: \( \neg \forall x \ p \equiv \exists x \ \neg p \), \( \neg \exists x \ p \equiv \forall x \ \neg p \)

\[ \forall x \left[ \exists y \ \neg \left( \neg Animal(y) \lor Loves(x,y) \right) \right] \lor \exists y \ Loves(y,x) \]
\[ \forall x \left[ \exists y \ \neg Animal(y) \land \neg Loves(x,y) \right] \lor \exists y \ Loves(y,x) \]
\[ \forall x \left[ \exists y \ Animal(y) \land \neg Loves(x,y) \right] \lor \exists y \ Loves(y,x) \]
Conversion to CNF contd.

3. Standardize variables:
   each quantifier should use a different one

   \[ \forall x \left[ \exists y \ Animal(y) \land \neg Loves(x, y) \right] \lor \left[ \exists z \ Loves(z, x) \right] \]

4. Skolemize: a more general form of existential instantiation.
   Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables:

   \[ \forall x \left[ Animal(F(x)) \land \neg Loves(x, F(x)) \right] \lor Loves(G(x), x) \]

   (reason: animal y could be a different animal for each x.)
Conversion to CNF contd.

5. Drop universal quantifiers:

\[ \text{Animal}(F(x)) \land \neg \text{Loves}(x,F(x))] \lor \text{Loves}(G(x),x) \]

(all remaining variables assumed to be universally quantified)

6. Distribute \lor over \land:

\[ \text{Animal}(F(x)) \lor \text{Loves}(G(x),x) \land \neg \text{Loves}(x,F(x)) \lor \text{Loves}(G(x),x) \]

Original sentence is now in CNF form – can apply same ideas to all sentences in KB to convert into CNF

Also need to include negated query

Then use resolution to attempt to derive the empty clause which show that the query is entailed by the KB
... it is a crime for an American to sell weapons to hostile nations:
\[ \text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x,y,z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x) \]

Nono … has some missiles, i.e., \( \exists x \) \( \text{Owns}(\text{Nono},x) \land \text{Missile}(x) \):
\[ \text{Owns}(\text{Nono},M_1) \text{ and } \text{Missile}(M_1) \]

... all of its missiles were sold to it by Colonel West
\[ \text{Missile}(x) \land \text{Owns}(\text{Nono},x) \Rightarrow \text{Sells}(\text{West},x,\text{Nono}) \]

Missiles are weapons:
\[ \text{Missile}(x) \Rightarrow \text{Weapon}(x) \]

An enemy of America counts as "hostile“:
\[ \text{Enemy}(x,\text{America}) \Rightarrow \text{Hostile}(x) \]

West, who is American ...
\[ \text{American}(\text{West}) \]

The country Nono, an enemy of America ...
\[ \text{Enemy}(\text{Nono},\text{America}) \]
Resolution proof

\[ \neg \text{American}(x) \lor \neg \text{Weapon}(y) \lor \neg \text{Sells}(x,y,z) \lor \neg \text{Hostile}(z) \lor \text{Criminal}(x) \]

[Diagram showing a resolution proof with clauses involving American, Weapon, Sells, Hostile, and Criminal.]
Second Example

**KB:**

Everyone who loves all animals is loved by someone
Anyone who kills animals is loved by no-one
Jack loves all animals
Either Curiosity or Jack killed the cat, who is named Tuna

**Query:** Did Curiosity kill the cat?

**Inference Procedure:**

Express sentences in FOL
Convert to CNF form and negated query
Resolution-based Inference

Confusing because the sentences have not been standardized apart…
Other Types of Reasoning (all unsound, often useful)

- **Inductive Reasoning (Induction)**
  - Reason from a set of examples to the general principle.
  - Fact: You`ve liked all movies starring Meryl Streep.
    Inference: You'll like her next movie.
  - Basis for most learning and scientific reasoning.

- **Abductive Reasoning (Abduction)**
  - Reason from facts to the conclusion that best explains them.
  - Fact: A large amount of black smoke is coming from a home.
    Abduction1: The house is on fire.
    Abduction2: Bad cook.
  - Basis for most debugging and medical diagnosis.

- **Analogical Reasoning (Analogy)**
  - Reason from known (source) to unknown (target).
  - Fact: Water flow in a hose; pressure, constrictions.
    Inference: Electricity flow in a circuit; voltage, resistance.
  - Basis for much teaching.
Modal Logic Examples

- □ represents **Necessary**
  - Analogous to “For All”
- ◇ represents **Possible**
  - Analogous to “There Exists”

- □ ↔ ¬◇¬ (Analogous to DeMorgan’s Law for Quantifiers)
- ◇ ↔ ¬□¬

- “It is *possible* that it will rain today.” ◇ RainToday
  “It is *not necessary* that it will *not* rain today.” ¬□¬ RainToday

- Modal Logic of Knowledge and Belief.
  - □ represents “x *knows* that ...”
  - ◇ represents “for all x knows, it may be true that ...”
    - Equivalently, “x does *not know* that it is *not* true that ...”
    - For reasoning about what other agents know and believe.

- Temporal Modal Logic
  - Modal operators [F] and [P] represent "henceforth" and "hitherto".
  - For reasoning about what will be and what has been.
Summary

- **Inference in FOL**
  - Simple approach: reduce all sentences to PL and apply propositional inference techniques
  - Generally inefficient

- **FOL inference techniques**
  - Unification
  - Generalized Modus Ponens
    - Forward-chaining
    - Backward-chaining
  - Resolution-based inference
    - Refutation-complete