Problem 1

Solution:

\[ P(\text{guessing the password in one trial}) = \left(\frac{1}{26}\right)^6 \]
\[ P(\text{guessing the password in a 1000 trials}) = 1000 \times \left(\frac{1}{26}\right)^6 \]
\[ P(\text{guessing the password in one trial, if I know that the first 4 positions must be vowels and the last two consonants}) = \left(\frac{1}{5}\right)^4 \times \left(\frac{1}{21}\right)^2 \]
\[ P(\text{guessing the password in a 1000 trials, if I know that the first 4 positions must be vowels and the last two consonants}) = 1000 \times \left(\frac{1}{5}\right)^4 \times \left(\frac{1}{21}\right)^2 \]

Problem 2

Solution:

\[ P(\text{still}) \times P(\text{walk}) \times P(\text{run}) = 0.6 \times 0.1 \times 0.3 = 0.018 \]

Permutation \([P(\text{still}), P(\text{walk}), P(\text{run})]\) = 6 \times (0.6 \times 0.1 \times 0.3) = 0.108

\[ P(\text{still})^5 = (0.6)^5 \]
\[ 1 - P(\text{still})^{10} = 1 - (0.6)^{10} \]

Problem 3

Solution: Let \(X\) denote the random variable that is defined as the sum of two fair dice. Then, there are 36 (equally likely) outcomes:

\[ P(X = 2) = P(X = 12) = 1/36 \]
\[ P(X = 3) = P(X = 11) = 2/36 \]
\[ P(X = 4) = P(X = 10) = 3/36 \]
\[ P(X = 5) = P(X = 9) = 4/36 \]
\[ P(X = 6) = P(X = 8) = 5/36 \]
\[ P(X = 7) = 6/36 \]
Problem 4

Solution: \( P(X = i) = \binom{6}{i} \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{6-i} \) Since the \( \left( \frac{1}{2} \right)^6 \) terms factor out, we merely have to find the maximum value for \( \binom{6}{i} \), which occurs for \( i = 3 \).

Note: An incorrect way to solve this problem is to say that the most likely value of \( X \) is the expected value of \( X \) \( (E[X] = np) \). This is not true in general. The most likely value is known as the mode, and is not in general the same as the mean (or the expected, or average) value. For example, \( E[X] = np \) need not be an integer in general (so it may equal any particular value of \( X \)), whereas the mode of \( X \) is the particular value of \( X \) that has the largest probability.

Problem 5

Solution: A useful way to solve this problem is to let \( X \) be a random variable that represents how many guesses the user makes before finding an available code. \( X \) takes values in the set \( \{1, 2, 3, \ldots, \infty\} \) and has a geometric distribution with probability \( p = \frac{k}{n} \) and \( n = 1 \) million. If you are not sure why \( X \) is geometric you may want to review the definition of a geometric random variable.

Thus, the probability of guessing on the first try is \( p \). The probability of having to guess 3 or more times is \( 1 - (P(X = 1) + P(X = 2)) \) where \( P(X = 1) = p \) and \( P(X = 2) = p(1 - p) \). The expected value is the expected value of a geometric random variable, which was shown (e.g., in the text) to be \( \frac{1}{p} \). The specific numbers are as follows:

1. for \( N=1,000,000 \), \( k=100,000 \)
   \[ P(X=1) = p = \frac{9}{100} \]
   \[ P(X\geq3) = 1 - (P(X=1)+P(X=2)) = 1 - \left( \frac{99}{100} \right) = \frac{1}{100} \]
   \[ E[X] = \frac{1}{p} = \frac{10}{9} \]

2. for \( N=1,000,000 \), \( k=500,000 \)
   \[ P(X=1) = p = \frac{1}{2} \]
   \[ P(X\geq3) = 1 - (P(X=1)+P(X=2)) = 1 - \left( \frac{1}{2} + \frac{1}{4} \right) = \frac{1}{4} \]
   \[ E[X] = \frac{1}{p} = 2 \]

3. for \( N=1,000,000 \), \( k=900,000 \)
   \[ P(X=1) = p = \frac{1}{10} \]
   \[ P(X\geq3) = 1 - (P(X=1)+P(X=2)) = 1 - \left( \frac{10}{100} + \frac{9}{100} \right) = \frac{81}{100} \]
   \[ E[X] = \frac{1}{p} = 10 \]

4. for \( N=1,000,000 \), \( k=990,000 \)
   \[ P(X=1) = p = \frac{1}{100} \]
\[ P(X \geq 3) = 1 - (P(X=1)+P(X=2)) = 1 - \left( \frac{100}{10000} + \frac{99}{10000} \right) = \frac{9801}{10000} \]

\[ E[X] = \frac{1}{p} = 100 \]

**Problem 6**

**Solution:** See Figures 1, 2, and 3.

1. For \( n = 10, p = 0.5 \) the probability mass function (pmf) is well distributed over all possible values of \( i \). The function is also symmetric in \( i \).

2. When \( n \) is increased to 100, the symmetry remains, the function is still centered at 0.5, but now the pmf is concentrated around 0.5. This implies that events with very few or many successes are relatively rare.

3. When we change \( p \) to 0.9, we shift the center of the pmf to \( np = 900 \).

![Figure 1: A bar graph of the binomial distribution with \( n = 10 \) and \( p = 0.5 \).](image1)

![Figure 2: A graph of the binomial distribution with \( n = 1000 \) and \( p = 0.5 \).](image2)
Problem 7
Solution:

1. We know the geometric pmf is \( P(k) = P\{X = k\} = (1 - p)^{k-1}p, k = 1, 2, \ldots \)
   
   Thus, the geometric cdf can be written as
   
   \[
   F(n) = P\{X \leq n\} = \sum_{k=1}^{n} P\{X = k\} = \sum_{k=1}^{n} (1 - p)^{k-1}p = p \sum_{k=1}^{n} (1 - p)^{k-1}
   \]
   
   \[
   = p \times \frac{(1 - (1 - p)^n)}{p} = 1 - (1 - p)^n
   \]

   How did we calculate \( S = \sum_{k=1}^{n} (1 - p)^{k-1} \) above?

   \[
   S = 1 + (1 - p) + (1 - p)^2 + (1 - p)^3 + \ldots + (1 - p)^{n-1} \tag{1}
   \]

   We get equation (2) by multiplying \((1 - p)\) on both sides of equation (1):

   \[
   (1 - p)S = (1 - p) + (1 - p)^2 + (1 - p)^3 + (1 - p)^4 \ldots + (1 - p)^n \tag{2}
   \]

   \[
   (1 - 2) \Rightarrow (1 - (1 - p))S = 1 - (1 - p)^n
   \]

   \[
   \Rightarrow S = \frac{1 - (1 - p)^n}{p}
   \]

2. Figure 4 shows the cumulative distribution function (cdf) for the geometric model.

3. Figure 5 shows the pmf for the geometric model. We see that for \( p = 0.9 \), the geometric distribution decays very quickly, since there is a relatively high probability (0.9) of “halting” at each point. Conversely, for \( p = 0.1 \), we see that it decays rather slowly. And \( p = 0.5 \) is in between. We also see that on a log-scale that the geometric distribution is a linear function of \( k \), which you can verify simply by taking the log of the pmf.
A. cdf of the geometric model  
B. Close-up of the cdf in the range k=1 to 10

Figure 4: Plots of the cumulative distribution function for the geometric model.

A. Normal scale in the range from k=1 to 10  
B. Log scale

Figure 5: Plots of the probability mass function for the geometric model.