40. Suppose that we look at the ten groups of integers in three consecutive locations around the circle (first-second-third, second-third-fourth, ..., eighth-ninth-tenth, ninth-tenth-first, and tenth-first-second). Since each number from 1 to 10 gets used three times in these groups, the sum of the sums of the ten groups must equal three times the sum of the numbers from 1 to 10, namely $3 \cdot 55 = 165$. Therefore the average sum is $165/10 = 16.5$. By Exercise 39, at least one of the sums must be greater than or equal to 16.5, and since the sums are whole numbers, this means that at least one of the sums must be greater than or equal to 17.

42. We show that each of these is equivalent to the statement (v) n is odd, say $n = 2k+1$. Example 1 showed that (v) implies (i), and Example 8 showed that (i) implies (v). For $(v) \to (ii)$ we see that $1 - n = 1 - (2k+1) = 2(-k)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $1 - n = 1 - 2m = 2(-m) + 1$, so $1 - n$ would be odd, and this completes the proof by contraposition that $(ii) \to (v)$. For $(v) \to (iii)$, we see that $n^3 = (2k+1)^3 = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$ is odd. Conversely, if n were even, say $n = 2m$, then we would have $n^3 = 2(4m^3)$, so n^3 would be even, and this completes the proof by contraposition that $(iii) \to (v)$. Finally, for $(v) \to (iv)$, we see that $n^2 + 1 = (2k+1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $n^2 + 1 = 2(2m^2) + 1$, so $n^2 + 1$ would be odd, and this completes the proof by contraposition that $(iv) \to (v)$.

SECTION 1.7 Proof Methods and Strategy

2. The cubes that might go into the sum are 1, 8, 27, 64, 125, 216, 343, 512, and 729. We must show that no two of these sum to a number on this list. If we try the 45 combinations $(1+1,\ 1+8,\ \ldots,\ 1+729,\ 8+8,\ 8+27,\ \ldots\ 8+729,\ \ldots,\ 729+729)$, we see that none of them works. Having exhausted the possibilities, we conclude that no cube less than 1000 is the sum of two cubes.

4. There are three main cases, depending on which of the three numbers is smallest. If a is smallest (or tied for smallest), then clearly $a \le \min(b,c)$, and so the left-hand side equals a. On the other hand, for the right-hand side we have $\min(a,c) = a$ as well. In the second case, b is smallest (or tied for smallest). The same reasoning shows us that the right-hand side equals b; and the left-hand side is $\min(a,b) = b$ as well. In the final case, in which c is smallest (or tied for smallest), the left-hand side is $\min(a,c) = c$, whereas the right-hand side is clearly also c. Since one of the three has to be smallest we have taken care of all the cases.

6. The number 1 has this property, since the only positive integer not exceeding 1 is 1 itself, and therefore the sum is 1. This is a constructive proof.

8. The only perfect squares that differ by 1 are 0 and 1. Therefore these two consecutive integers cannot both be perfect squares. This is a nonconstructive proof—we do not know which of them meets the requirement. (In fact, a computer algebra system will tell us that neither of them is a perfect square.)

10. Of these three numbers, at least two must have the same sign (both positive or both negative), since there are only two signs. (It is conceivable that some of them are zero, but we view zero as positive for the purposes of this problem.) The product of two with the same sign is nonnegative. This was a nonconstructive proof, since we have not identified which product is nonnegative. (In fact, a computer algebra system will tell us that all three are positive, so all three products are positive.)

12. An assertion like this one is implicitly universally quantified—it means that *for all* rational numbers a and b, a^b is rational. To disprove such a statement it suffices to provide one counterexample. Take $a = 2$ and $b = 1/2$. Then $a^b = 2^{1/2} = \sqrt{2}$, and we know from Example 10 in Section 1.6 that $\sqrt{2}$ is not rational.

14. We know from algebra that the following equations are equivalent: $ax + b = c$, $ax = c - b$. $x = (c-b)/a$. This shows, constructively, what the unique solution of the given equation is.

16. Given r, let a be the closest integer to r less than r, and let b be the closest integer to r greater than r. In the notation to be introduced in Section 2.3, $a = \lfloor r \rfloor$ and $b = \lceil r \rceil$. In fact, $b = a + 1$. Clearly the distance between r and any integer other than a or b is greater than 1 so cannot be less than $1/2$. Furthermore, since r is irrational, it cannot be exactly half-way between a and b, so exactly one of $r - a < 1/2$ and $b - r < 1/2$ holds.

18. Given x, let n be the greatest integer less than or equal to x, and let $\epsilon = x - n$. In the notation to be introduced in Section 2.3, $n = \lfloor x \rfloor$. Clearly $0 \le \epsilon < 1$, and ϵ is unique for this n. Any other choice of n would cause the required ϵ to be less than 0 or greater than or equal to 1, so n is unique as well.

20. We follow the hint. The square of every real number is nonnegative, so $(x - 1/x)^2 \ge 0$. Multiplying this out and simplifying, we obtain $x^2 - 2 + 1/x^2 \ge 0$, so $x^2 + 1/x^2 \ge 2$, as desired.

22. If $a = 5$ and $b = 8$, then the quadratic mean is $\sqrt{(5^2+8^2)/2} \approx 6.67$, and the arithmetic mean is $(5+8)/2 = 6.5$. If $a = 10$ and $b = 100$, then the quadratic mean is $\sqrt{(10^2+100^2)/2} \approx 71.06$, and the arithmetic mean is $(10+100)/2 = 55$. We conjecture that the quadratic mean of a and b is always greater than their arithmetic mean if a and b are distinct positive real numbers (clearly if $a = b$ then both means are this common value). So we want to verify the inequality $\sqrt{(a^2+b^2)/2} > (a+b)/2$. Squaring both sides (this is legal because everything in sight is positive) and multiplying by 4 gives us the equivalent inequality $2a^2 + 2b^2 > a^2 + 2ab + b^2$, which is in turn equivalent to $(a-b)^2 > 0$ after putting everything on the left-hand side and factoring. This is clearly always true, and our proof is complete.

24. If we were to end up with nine 0's, then in the step before this we must have had either nine 0's or nine 1's, since each adjacent pair of bits must have been equal and therefore all the bits must have been the same. Thus if we are to start with something other than nine 0's and yet end up with nine 0's, we must have had nine 1's at some point. But in the step before that each adjacent pair of bits must have been different; in other words, they must have alternated 0, 1, 0, 1, and so on. This is impossible with an odd number of bits. This contradiction shows that we can never get nine 0's.

26. Clearly only the last two digits of n contribute to the last two digits of n^2. So we can compute 0^2, 1^2, 2^2, 3^2, ..., 99^2, and record the last two digits, omitting repetitions. We obtain 00, 01, 04, 09, 16, 25, 36, 49, 64, 81, 21, 44, 69, 96, 56, 89, 24, 61, 41, 84, 29, 76. From that point on, the list repeats in reverse order (as we take the squares from 25^2 to 49^2, and then it all repeats again as we take the squares from 50^2 to 99^2). The reason for these last two statements are that $(50-n)^2 = 2500 - 100n + n^2$, so $(50-n)^2$ and n^2 have the same two final digits, and $(50+n)^2 = 2500 + 100n + n^2$, so $(50+n)^2$ and n^2 have the same two final digits. Thus our list (which contains 22 numbers) is complete.

28. If $|y| \ge 2$, then $2x^2 + 5y^2 \ge 2x^2 + 20 \ge 20$, so the only possible values of y to try are 0 and ± 1. In the former case we would be looking for solutions to $2x^2 = 14$ and in the latter case to $2x^2 = 9$. Clearly there are no integer solutions to these equations, so there are no solutions to the original equation.

30. Following the hint, we let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$. Then $x^2 + y^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 = z^2$. Thus we have found infinitely many solutions, since m and n can be arbitrarily large.

32. One proof that $\sqrt[3]{2}$ is irrational is similar to the proof that $\sqrt{2}$ is irrational, given in Example 10 in Section 1.6. It is a proof by contradiction. Suppose that $2^{1/3}$ (or $\sqrt[3]{2}$, which is the same thing) is the rational number p/q, where p and q are positive integers with no common factors (the fraction is in lowest terms). Cubing, we see that $2 = p^3/q^3$, or, equivalently, $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd numbers is odd, this means that p is even, so we can write $p = 2s$. Substituting into the equation $p^3 = 2q^3$, we obtain $8s^3 = 2q^3$, which simplifies to $4s^3 = q^3$.

Now we play the same game with q. Since q^3 is even, q must be even. We have now concluded that p and q are both even, that is, that 2 is a common divisor of p and q. This contradicts the choice of p/q to be in lowest terms. Therefore our original assumption—that $\sqrt[3]{2}$ is rational—is in error, so we have proved that $\sqrt[3]{2}$ is irrational.

34. The average of two different numbers is certainly always between the two numbers. Furthermore, the average a of rational number x and irrational number y must be irrational, because the equation $a = (x+y)/2$ leads to $y = 2a - x$, which would be rational if a were rational.

36. The solution is not unique, but here is one way to measure out four gallons. Fill the 5-gallon jug from the 8-gallon jug, leaving the contents $(3,5,0)$, where we are using the ordered triple to record the amount of water in the 8-gallon jug, the 5-gallon jug, and the 3-gallon jug, respectively. Next fill the 3-gallon jug from the 5-gallon jug, leaving $(3,2,3)$. Pour the contents of the 3-gallon jug back into the 8-gallon jug, leaving $(6,2,0)$. Empty the 5-gallon jug's contents into the 3-gallon jug, leaving $(6,0,2)$, and then fill the 5-gallon jug from the 8-gallon jug, producing $(1,5,2)$. Finally, top off the 3-gallon jug from the 5-gallon jug, and we'll have $(1,4,3)$, with four gallons in the 5-gallon jug.

38. a) $16 \to 8 \to 4 \to 2 \to 1$
b) $11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$
c) $35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$
d) $113 \to 340 \to 170 \to 85 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$

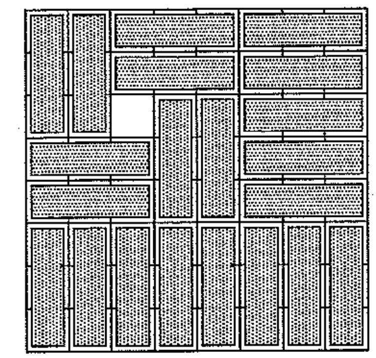
40. This is easily done, by laying the dominoes horizontally, three in the first and last rows and four in each of the other six rows.

42. Without loss of generality, we number the squares from 1 to 25, starting in the top row and proceeding left to right in each row; and we assume that squares 5 (upper right corner), 21 (lower left corner), and 25 (lower right corner) are the missing ones. We argue that there is no way to cover the remaining squares with dominoes.

By symmetry we can assume that there is a domino placed in 1-2 (using the obvious notation). If square 3 is covered by 3-8, then the following dominoes are forced in turn: 4-9, 10-15, 19-20, 23-24, 17-22, and 13-18, and now no domino can cover square 14. Therefore we must use 3-4 along with 1-2. If we use all of 17-22, 18-23, and 19-24, then we are again quickly forced into a sequence of placements that lead to a contradiction. Therefore without loss of generality, we can assume that we use 22-23, which then forces 19-24, 15-20, 9-10, 13-14, 7-8, 6-11, and 12-17, and we are stuck once again. This completes the proof by contradiction that no placement is possible.

44. The barriers shown in the diagram split the board into one continuous closed path of 64 squares, each adjacent to the next (for example, start at the upper left corner, go all the way to the right, then all the way down, then all the way to the left, and then weave your way back up to the starting point). Because each square in the path is adjacent to its neighbors, the colors alternate. Therefore, if we remove one black square and one white square, this closed path decomposes into two paths, each of which starts in one color and ends in the other color (and therefore has even length). Clearly each such path can be covered by dominoes by starting at one end. This completes the proof.

46. If we study Figure 7, we see that by rotating or reflecting the board, we can make any square we wish nonwhite, with the exception of the squares with coordinates $(3,3)$, $(3,6)$, $(6,3)$, and $(6,6)$. Therefore the same argument as was used in Example 22 shows that we cannot tile the board using straight triominoes if any one of those other 60 squares is removed. The following drawing (rotated as necessary) shows that we can tile the board using straight triominoes if one of those four squares is removed.



48. We will use a coloring of the 10×10 board with four colors as the basis for a proof by contradiction showing that no such tiling exists. Assume that 25 straight tetrominoes can cover the board. Some will be placed horizontally and some vertically. Because there is an odd number of tiles, the number placed horizontally and the number placed vertically cannot both be odd, so assume without loss of generality that an even number of tiles are placed horizontally. Color the squares in order using the colors red, blue, green, yellow in that order repeatedly, starting in the upper left corner and proceeding row by row, from left to right in each row. Then it is clear that every horizontally placed tile covers one square of each color and each vertically placed tile covers either zero or two squares of each color. It follows that in this tiling an even number of squares of each color are covered. But this contradicts the fact that there are 25 squares of each color. Therefore no such coloring exists.

SUPPLEMENTARY EXERCISES FOR CHAPTER 1

2. The truth table is as follows.

p	q	r	$p \vee q$	$p \wedge \neg r$	$(p \vee q) \to (p \wedge \neg r)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	F	T

4. a) The converse is "If I drive to work today, then it will rain." The contrapositive is "If I do not drive to work today, then it will not rain." The inverse is "If it does not rain today, then I will not drive to work."
b) The converse is "If $x \geq 0$ then $|x| = x$." The contrapositive is "If $x < 0$ then $|x| \neq x$." The inverse is "If $|x| \neq x$, then $x < 0$."
c) The converse is "If n^2 is greater than 9, then n is greater than 3." The contrapositive is "If n^2 is not greater than 9, then n is not greater than 3." The inverse is "If n is not greater than 3, then n^2 is not greater than 9."