CHAPTER 11
Boolean Algebra

SECTION 11.1 Boolean Functions

2. **a)** Since $x \cdot 1 = x$, the only solution is $x = 0$.

b) Since $0 + 0 = 0$ and $1 + 1 = 1$, the only solution is $x = 0$.

c) Since this equation holds for all x, there are two solutions, $x = 0$ and $x = 1$.

d) Since either x or $\overline{x}$ must be 0, no matter what x is, there are no solutions.

4. **a)** We compute $(\overline{1} \cdot \overline{0}) + (1 \cdot \overline{0}) = (0 \cdot 1) + (1 \cdot 1) = 0 + 1 = 1$.

b) Following the instructions, we have $(\neg \mathbf{T} \wedge \neg \mathbf{F}) \vee (\mathbf{T} \wedge \neg \mathbf{F}) \equiv \mathbf{T}$.

6. In each case, we compute the various components of the final expression and put them together as indicated. For part **(a)** we have simply

x	y	z	$\overline{z}$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

For part **(b)** we have

x	y	z	$\overline{x}$	$\overline{x}\,y$	$\overline{y}$	$\overline{y}\,z$	$\overline{x}\,y + \overline{y}\,z$
1	1	1	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	1	1	1
1	0	0	0	0	1	0	0
0	1	1	1	1	0	0	1
0	1	0	1	1	0	0	1
0	0	1	1	0	1	1	1
0	0	0	1	0	1	0	0

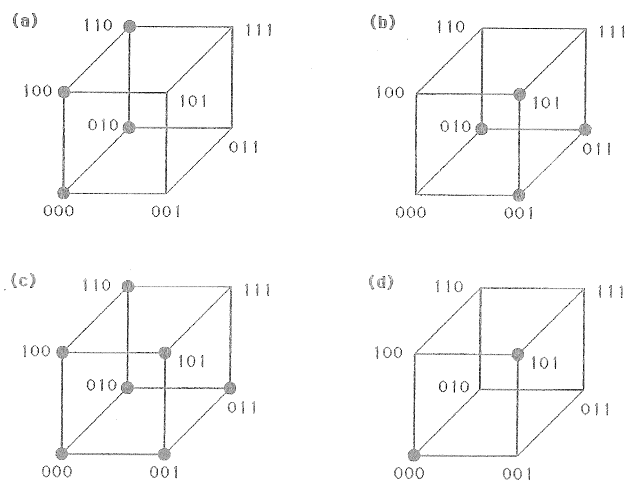
For part **(c)** we have

x	y	z	$\overline{y}$	$x\,\overline{y}\,z$	xyz	$\overline{xyz}$	$x\,\overline{y}\,z + \overline{xyz}$
1	1	1	0	0	1	0	0
1	1	0	0	0	0	1	1
1	0	1	1	1	0	1	1
1	0	0	1	0	0	1	1
0	1	1	0	0	0	1	1
0	1	0	0	0	0	1	1
0	0	1	1	0	0	1	1
0	0	0	1	0	0	1	1

For part (d) we have

x	y	z	$\overline{x}$	$\overline{y}$	$\overline{z}$	xz	$\overline{x}\,\overline{z}$	$xz+\overline{x}\,\overline{z}$	$\overline{y}(xz+\overline{x}\,\overline{z})$
1	1	1	0	0	0	1	0	1	0
1	1	0	0	0	1	0	0	0	0
1	0	1	0	1	0	1	0	1	1
1	0	0	0	1	1	0	0	0	0
0	1	1	1	0	0	0	0	0	0
0	1	0	1	0	1	0	1	1	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	1	1	0	1	1	1

8. In each case, we note from our solution to Exercise 6 which vertices need to be blackened in the cube, as in Figure 1.



10. There are 2^{2^n} different Boolean functions of degree n, so the answer is $2^{2^7} = 2^{128} \approx 3.4 \times 10^{38}$.

12. The only way for the sum to have the value 1 is for one of the summands to have the value 1, since $0+0+0=0$. Each summand is 1 if and only if the two variables in the product making up that summand are both 1. The conclusion follows.

14. If $x = 0$, then $\overline{\overline{x}} = \overline{\overline{0}} = \overline{1} = 0 = x$. We obtain $\overline{\overline{1}} = 1$ by a similar calculation. The relevant table, exhibiting this calculation, has only two rows.

16. We just plug in $x = 0$ and $x = 1$ and see that the equations hold in each case. The relevant tables, exhibiting these calculations, have only two rows.

18. We can make a table to list the four possible combinations of values for x and y in each case, and check that $x + y = y + x$ and $xy = yx$. Alternatively, we simply note that $x + y = 0$ if and only if $x = y = 0$, and $xy = 1$ if and only if $x = y = 1$, and these statement are symmetric in the variables x and y.

20. We can make a table to list all the possibilities, but instead let us argue more directly. The left-hand side of this equation is 1 precisely when either $x = 1$ or both y and z are 1. In the former case, both $x + y$ and $x + z$ are 1, so their product is 1, and in the latter case both $x + y$ and $x + z$ are 1, so again their product is 1. Conversely, the left-hand side is 0 when $x = 0$ and at least one of y and z is 0. In this case, at least one of $x + y$ and $x + z$ is 0, so their product is 0.

22. The unit property states that $x + \overline{x} = 1$. There are only two things to check: $0 + \overline{0} = 0 + 1 = 1$ and $1 + \overline{1} = 1 + 0 = 1$. The relevant table, exhibiting this calculation, has only two rows.

24. a) Since $0 \oplus 0 = 0$ and $1 \oplus 0 = 1$, this expression simplifies to x.
b) Since $0 \oplus 1 = 1$ and $1 \oplus 1 = 0$, this expression simplifies to $\overline{x}$.
c) Looking at the definition, we see that $x \oplus x = 0$ for all x.
d) This is similar to part (c); this time the expression always equals 1.

26. A glance at the definition shows that $x \oplus y = y \oplus x$ for all four possibilities for x and y.

28. In each case we simply change each 0 to a 1 and vice versa, and change all the sums to products and vice versa.
a) xy b) $\overline{x} + \overline{y}$ c) $(x + y + z)(\overline{x} + \overline{y} + \overline{z})$ d) $(x + \overline{z})(x + 1)(\overline{x} + 0)$

30. By Exercise 29, what we are asked to show is equivalent to the statement that for all values of $x_1, x_2, \ldots, x_n$, we have $\overline{F(\overline{x}_1, \ldots, \overline{x}_n)} = \overline{G(\overline{x}_1, \ldots, \overline{x}_n)}$. Now this is clearly equivalent to $F(\overline{x}_1, \ldots, \overline{x}_n) = G(\overline{x}_1, \ldots, \overline{x}_n)$. But the value of the n-tuple $(\overline{x}_1, \ldots, \overline{x}_n)$ ranges over all n-tuples of 0's and 1's as the value of $(x_1, \ldots, x_n)$ ranges over all n-tuples of 0's and 1's (albeit in a different order). Since we are given that $F = G$, the desired conclusion follows.

32. Suppose that you specify $F(0,0,0)$. Then the equations determine $F(\overline{0}, \overline{0}, 0) = F(1,1,0)$ and $F(\overline{0}, 0, \overline{0}) = F(1,0,1)$. It also therefore determines $F(\overline{1}, 1, \overline{0}) = F(0,1,1)$, but nothing else. If we now also specify $F(1,1,1)$ (and there are no restrictions imposed so far), then the equations tell us, in a similar way, what $F(0,0,1)$, $F(0,1,0)$, and $F(1,0,0)$ are. This completes the definition of F. Since we had two choices in specifying $F(0,0,0)$ and two choices in specifying $F(1,1,1)$, the answer is $2 \cdot 2 = 4$.

34. We need to replace each 0 by $\mathbf{F}$, 1 by $\mathbf{T}$, $+$ by $\vee$, $\cdot$ (or Boolean product implied by juxtaposition) by $\wedge$, and $^{-}$ by $\neg$. We also replace x by p and y by q so that the variables look like they represent propositions, and we replace the equals sign by the logical equivalence symbol. We also add parentheses for clarification. Thus for the first absorption law in Table 5, $x + xy = x$ becomes $p \vee (p \wedge q) \equiv p$, which is the first absorption law in Table 6 of Section 1.2. Dually, $x(x + y) = x$ becomes $p \wedge (p \vee q) \equiv p$ for the other absorption law.

36. To prove that the complement of x is unique, we suppose that y is a complement (i.e., $x \vee y = 1$ and $x \wedge y = 0$) and play with the symbols (using the axioms in Definition 1) until we have $y = \overline{x}$. The reason for each step in this proof is just one (or more) of these axioms.

$$\begin{aligned} y = y \wedge 1 &= y \wedge (x \vee \overline{x}) \\ &= (y \wedge x) \vee (y \wedge \overline{x}) \\ &= (x \wedge y) \vee (y \wedge \overline{x}) \\ &= 0 \vee (y \wedge \overline{x}) \\ &= y \wedge \overline{x} \\ &= (y \wedge \overline{x}) \vee 0 \\ &= (y \wedge \overline{x}) \vee (x \wedge \overline{x}) \\ &= (\overline{x} \wedge y) \vee (\overline{x} \wedge x) \\ &= \overline{x} \wedge (y \vee x) \\ &= \overline{x} \wedge (x \vee y) \\ &= \overline{x} \wedge 1 = \overline{x} \end{aligned}$$

38. This follows from Exercise 36, where we showed that the complement of an element z is that unique element y such that $z \vee y = 1$ and $z \wedge y = 0$. For this exercise, we just need to show that $y = x$ fits this definition if we choose $z = \overline{x}$. In other words, this will show that x is the complement of $\overline{x}$. But plugging into our equations we have simply $\overline{x} \vee x = 1$ and $\overline{x} \wedge x = 0$, which follow from the axioms (including commutativity).

40. We start with the left-hand side and try to obtain the right-hand side. We freely use the axioms from Definition 1 as well as the result in Exercise 35. For the first identity,

$$\begin{aligned} x \wedge (y \vee (x \wedge z)) &= (x \wedge y) \vee (x \wedge x \wedge z) \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

The second proof is dual (interchange the roles of $\wedge$ and $\vee$).

42. Since all the axioms come in dual pairs, any proof of an identity can be transformed into a proof of the dual identity by interchanging $\vee$ with $\wedge$ and interchanging 0 with 1. Hence if an identity is valid, so is its dual.

SECTION 11.2 Representing Boolean Functions

2. **a)** We can rewrite this as $F(x,y) = \overline{x} \cdot 1 + \overline{y} \cdot 1 = \overline{x}(y + \overline{y}) + y(x + \overline{x})$. Expanding and using the commutative and idempotent laws, this simplifies to $\overline{x}\,y + \overline{x}\,\overline{y} + x\,y$.

b) This is already in sum-of-products form.

c) We need to write the sum of all products; the answer is $x\,y + x\,\overline{y} + \overline{x}\,y + \overline{x}\,\overline{y}$.

d) As in part (a), we have $F(x,y) = 1 \cdot \overline{y} = (x + \overline{x})y = x\,y + \overline{x}\,y$.

4. **a)** We need to write all the terms that have $\overline{x}$ in them. Thus the answer is $\overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

b) We need to write all the terms that include either $\overline{x}$ or $\overline{y}$. Thus the answer is $x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

c) We need to include all the terms that have both $\overline{x}$ and $\overline{y}$. Thus the answer is $\overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

d) We need to include all the terms that have at least one of $\overline{x}$, $\overline{y}$, and $\overline{z}$. This is all the terms except $x\,y\,z$, so the answer is $x\,y\,\overline{z} + x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,\overline{y}\,\overline{z}$.

6. We need to include all terms that have three or more of the variables in their uncomplemented form. This will give us a total of $1 + 5 + 10 = 16$ terms. The answer is

$$\begin{aligned} x_1\,x_2\,x_3\,x_4\,x_5 &+ x_1\,x_2\,x_3\,x_4\,\overline{x}_5 + x_1\,x_2\,x_3\,\overline{x}_4\,x_5 + x_1\,x_2\,\overline{x}_3\,x_4\,x_5 + x_1\,\overline{x}_2\,x_3\,x_4\,x_5 + \overline{x}_1\,x_2\,x_3\,x_4\,x_5 \\ &+ x_1\,x_2\,x_3\,\overline{x}_4\,\overline{x}_5 + x_1\,x_2\,\overline{x}_3\,x_4\,\overline{x}_5 + x_1\,x_2\,\overline{x}_3\,\overline{x}_4\,x_5 + x_1\,\overline{x}_2\,x_3\,x_4\,\overline{x}_5 + x_1\,\overline{x}_2\,x_3\,\overline{x}_4\,x_5 \\ &+ x_1\,\overline{x}_2\,\overline{x}_3\,x_4\,x_5 + \overline{x}_1\,x_2\,x_3\,x_4\,\overline{x}_5 + \overline{x}_1\,x_2\,x_3\,\overline{x}_4\,x_5 + \overline{x}_1\,x_2\,\overline{x}_3\,x_4\,x_5 + \overline{x}_1\,\overline{x}_2\,x_3\,x_4\,x_5\,. \end{aligned}$$

8. We follow the hint and form the product $(\overline{x} + \overline{y} + z)(x + y + z)(x + \overline{y} + z)$. It will have the value 0 as long as one of the factors has the value 0.

10. We follow the hint and include one maxterm in this product for each combination of variables for which the function has the value 0 (see Exercise 9). Since a product is 0 if and only if at least one of the factors is 0, this sum has the desired value.

12. We need to use De Morgan's law to replace each occurrence of $s + t$ by $\overline{(\overline{s}\,\overline{t})}$, simplifying by use of the double complement law if possible.

a) $(x + y) + z = \overline{(\overline{(x + y)}\,\overline{z})} = \overline{(\overline{x}\,\overline{y}\,\overline{z})}$ **b)** $x + \overline{y}\,(\overline{x} + z) = \overline{(\overline{x}\,\overline{(\overline{y}\,(\overline{x} + z))})} = \overline{(\overline{x}\,\overline{(\overline{y}\,\overline{(x\,\overline{z})})})}$

c) In this case we can just apply De Morgan's law directly, to obtain $\overline{x}\,\overline{\overline{y}} = \overline{x}\,y$.

d) The second factor is changed in a manner similar to part (a). Thus the answer is $\overline{x}\overline{(\overline{x}\,y\,z)}$.

14. **a)** We use the definition of $|$. If $x = 1$, then $x \mid x = 0$; and if $x = 0$, then $x \mid x = 1$. These are precisely the corresponding values of $\overline{x}$.

b) We can construct a table to look at all four cases, as follows. Since the fourth and fifth columns are equal, the expressions are equivalent.

x	y	$x \mid y$	$(x \mid y) \mid (x \mid y)$	xy
1	1	0	1	1
1	0	1	0	0
0	1	1	0	0
0	0	1	0	0

c) We can construct a table to look at all four cases, as follows. Since the fifth and sixth columns are equal, the expressions are equivalent.

x	y	$x \mid x$	$y \mid y$	$(x \mid x) \mid (y \mid y)$	$x + y$
1	1	0	0	1	1
1	0	0	1	1	1
0	1	1	0	1	1
0	0	1	1	0	0

16. Since we already know that complementation, sum and product together are functionally complete, and since Exercise 15 tells us how to write all of these operations totally in terms of $\downarrow$, we can write every Boolean function totally in terms of $\downarrow$.

18. We use the results of Exercise 15.

a) $(x + y) + z = ((x + y) \downarrow z) \downarrow ((x + y) \downarrow z) = (((x \downarrow y) \downarrow (x \downarrow y)) \downarrow z) \downarrow (((x \downarrow y) \downarrow (x \downarrow y)) \downarrow z)$

b) $(x + z)y = ((x + z) \downarrow (x + z)) \downarrow (y \downarrow y) = (((x \downarrow z) \downarrow (x \downarrow z)) \downarrow ((x \downarrow z) \downarrow (x \downarrow z))) \downarrow (y \downarrow y)$

c) This is already in the desired form, since it has no operators.

d) $x\overline{y} = (x \downarrow x) \downarrow (\overline{y} \downarrow \overline{y}) = (x \downarrow x) \downarrow ((y \downarrow y) \downarrow (y \downarrow y))$

20. We assume here that the constants 0 and 1 cannot be used (the answers to parts **(a)** and **(c)** are different if constants are allowed).

a) Note that $0 + 0 = 0 \oplus 0 = 0$. This means that every function that uses only these two operations must have the value 0 when the inputs are all 0. Therefore using only these two operations, we cannot construct the Boolean function that is 1 for all inputs.

b) This set is not functionally complete. Note first that $\overline{(x \oplus y)} = \overline{x} \oplus y$. Thus every expression involving these two operations and x and y can be reduced to an *XOR* of the literals x, $\overline{x}$, y, and $\overline{y}$. Note that $\oplus$ is commutative and associative, so that we can rearrange such expressions to group things conveniently. Also, since $x \oplus x = 0$, $x \oplus \overline{x} = 1$, $x \oplus 1 = \overline{x}$ and $x \oplus 0 = x$, and similarly for y (see Exercise 24 in Section 11.1), we can reduce all such expressions to one of the expressions 0, 1, x, y, $\overline{x}$, $\overline{y}$, $x \oplus y$, $x \oplus \overline{y}$, $\overline{x} \oplus y$, or $\overline{x} \oplus \overline{y}$. Since none of these has the same table of values as $x + y$, we conclude that the set is not functionally complete.

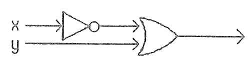
c) This is similar to part **(a)**. This time we note that $0 \cdot 0 = 0 \oplus 0 = 0$. Again this means that every function that uses only these two operations must have the value 0 when the inputs are all 0. Therefore using only these two operations, we cannot construct the Boolean function that is 1 for all inputs.

SECTION 11.3 Logic Gates

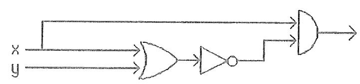
2. The inputs to the *AND* gate are $\overline{x}$ and $\overline{y}$. The output is then passed through the inverter. Therefore the final output is $\overline{(\overline{x}\,\overline{y})}$. Note that there is a simpler way to form a circuit equivalent to this one, namely $x+y$.

4. This is similar to the previous three exercises. The output is $\overline{(\overline{x}\,y\,z)}(\overline{x}+y+\overline{z})$.

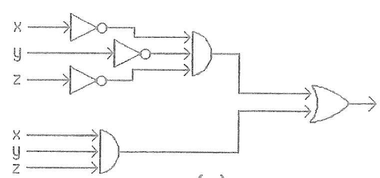
6. We build these circuits up exactly as the expressions are built up. In part (b), for example, we use an *AND* gate to join the outputs of the inverter (which was applied to the output of the *OR* gate applied to x and y) and x.



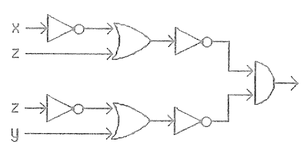
(a)



(b)

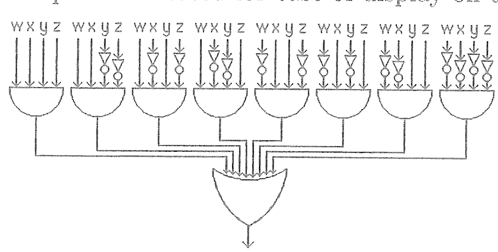


(c)

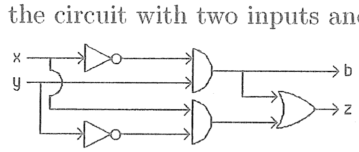


(d)

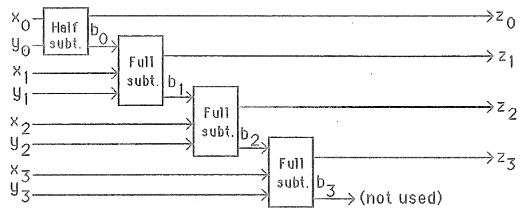
8. In analogy to the situation with three switches in Example 3, we write down the expression we want the circuit to implement: $w\,x\,y\,z + w\,x\,\overline{y}\,\overline{z} + w\,\overline{x}\,y\,\overline{z} + w\,\overline{x}\,\overline{y}\,z + \overline{w}\,x\,y\,\overline{z} + \overline{w}\,x\,\overline{y}\,z + \overline{w}\,\overline{x}\,y\,z + \overline{w}\,\overline{x}\,\overline{y}\,\overline{z}$. The circuit will have 32 inputs, combined by *AND* gates in groups of four, with inverters where necessary, to produce outputs corresponding to the eight minterms in this expression. These outputs are combined with one big *OR* gate. The circuit is shown below, with the picture rotated for ease of display on the page.



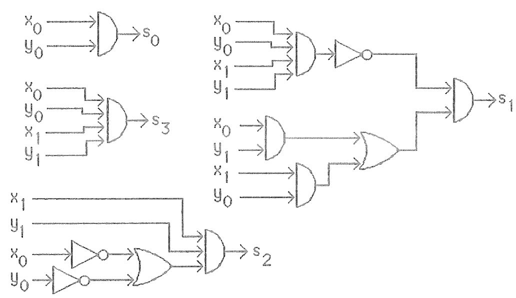
10. First we must determine what the outputs are to be. Let x and y be the input bits, where we want to compute $x-y$. There are two outputs: the difference bit z and the borrow bit b. The borrow will be 1 if a borrow is necessary, which happens only when $x=0$ and $y=1$. Thus $b=\overline{x}\,y$. The difference bit will be 1 when $x=1$ and $y=0$, and when $x=0$ and $y=1$; and it will be 0 in the cases in which $x=y$. Therefore we have $z=\overline{x}\,y+x\,\overline{y}$, which is the same as $b+x\,\overline{y}$. Thus we can draw the half subtractor as shown below. In analogy with Figure 8, we represent the circuit with two inputs and two outputs.



12. We need to combine half subtractors and full subtractors in much the same way that half adders and full adders were combined to produce a circuit to add binary numbers. The first bit of the answer (z_0) is the difference bit between the first two bits of the input (x_0 and y_0), obtained using the half subtractor. The borrow bit output from the half subtractor (b_0) is then the borrow bit input to the full subtractor for determining the second bit of the answer, and so on. Note that the final borrow b_3 must be 0 and is not used.



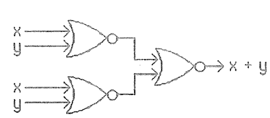
14. Let $(s_3 s_2 s_1 s_0)_2$ be the product. We need to write down Boolean expressions for each of these bits. Clearly $s_0 = x_0 y_0$. The bit s_1 is a 1 if one, but not both, of the products $x_0 y_1$ and $x_1 y_0$ are 1. Therefore we have $s_1 = (x_0 y_1 + x_1 y_0)\overline{(x_0 x_1 y_0 y_1)}$. A similar analysis will show that $s_2 = x_1 y_1 (\overline{x}_0 + \overline{y}_0)$, and that $s_3 = x_0 x_1 y_0 y_1$. The circuit we want has one circuit for each of these bits.



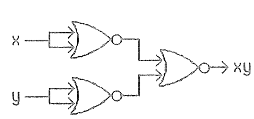
16. The answers here are duals to the answers for Exercise 15. Note that the usual symbol $\downarrow$ represents the *NOR* operation.

a) The circuit is the same as in Exercise 15a, with a *NOR* gate in place of a *NAND* gate, since $\overline{x} = x \mid x = x \downarrow x$.

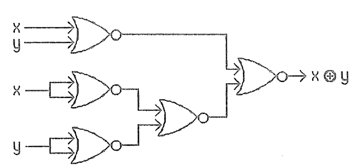
b) Since $x + y = (x \downarrow y) \downarrow (x \downarrow y)$, the answer is as shown.



c) Since $xy = (x \downarrow x) \downarrow (y \downarrow y)$, the answer is as shown.



d) We use the representation $x \oplus y = (x + y)\overline{(xy)} = \overline{(\overline{(x+y)} + x y)} = (x \downarrow y) \downarrow (xy) = (x \downarrow y) \downarrow ((x \downarrow x) \downarrow (y \downarrow y))$, obtaining the following circuit.



18. We know that the sum bit in the half adder is $s = x \oplus y = x\overline{y} + \overline{x}y$. The answer to Exercise 16d shows precisely this gate constructed from *NOR* gates, so it gives us this part of the answer. Also, the carry bit in the half adder is $c = xy$. The answer to Exercise 16c shows precisely this gate constructed from *NOR* gates, so it gives us this part of the answer.

20. **a)** The initial inputs have depth 0. Therefore the three *AND* gates all have depth 1, as do their outputs. Therefore the *OR* gate has depth 2, which is the depth of the circuit.

b) The *AND* gate at the top of Figure 6 and the two *NOT* gates have depth 1, so the *AND* gate at the bottom has depth 2. Therefore the inputs to the *OR* gate have depth 1 or 2, so its depth is 3 (one more than the maximum of these), which is the depth of the circuit.

c) The maximum of the depths of the gates is 3, for the final *AND* gate, since the *NOT* gate feeding it has depth 2. Therefore the depth of the circuit is 3.

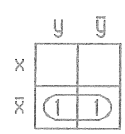
d) We have to be careful here, since the outputs of the half-adder are 3 for the sum but 1 for the carry. So the depth of the half adder at the top of this full adder is 6 for its sum output and 4 for its carry output. The carry output goes through one more gate, giving a total depth of 5 for the *OR* gate, but the depth of the circuit is 6, because of the output at the upper right.

SECTION 11.4 Minimization of Circuits

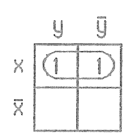
2. We just write down the minterms for which there is a 1 in the corresponding box, and join them with $+$.

a) $xy + \overline{x}y + \overline{x}\,\overline{y}$ **b)** $xy + x\overline{y}$ **c)** $xy + x\overline{y} + \overline{x}y + \overline{x}\,\overline{y}$

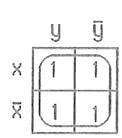
4. **a)** The K-map is shown here. The two 1's combine into the larger block representing the expression $\overline{x}$. Therefore the answer is $\overline{x}$.



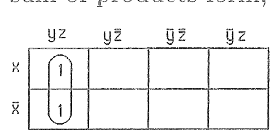
b) The K-map is as shown here. The two 1's combine into the larger block representing the expression x. Therefore the answer is x.



c) All four 1's combine to form the larger block which represents the term 1; this is the answer.



6. **a)** The function is already presented in its sum-of-products form, so we easily draw the following K-map.



The grouping shown here tells us that the simplest Boolean expression is just yz. Therefore the circuit shown below answers this exercise.