CHAPTER 2
Basic Structures: Sets, Functions, Sequences, and Sums

SECTION 2.1 Sets

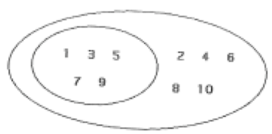
2. There are of course an infinite number of correct answers.
a) $\{3n \mid n = 0,1,2,3,4\}$ or $\{x \mid x$ is a multiple of $3 \wedge 0 \le x \le 12\}$.
b) $\{x \mid -3 \le x \le 3\}$, where we are assuming that the domain (universe of discourse) is the set of integers.
c) $\{x \mid x$ is a letter of the word *monopoly* other than l or $y\}$.

4. Each of the sets is a subset of itself. Aside from that, the only relations are $B \subseteq A$, $C \subseteq A$, and $C \subseteq D$.

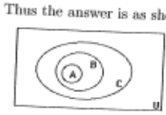
6. a) Since the set contains only integers and $\{2\}$ is a set, not an integer, $\{2\}$ is not an element.
b) Since the set contains only integers and $\{2\}$ is a set, not an integer, $\{2\}$ is not an element.
c) The set has two elements. One of them is patently $\{2\}$.
d) The set has two elements. One of them is patently $\{2\}$.
e) The set has two elements. One of them is patently $\{2\}$.
f) The set has only one element, $\{\{2\}\}$; since this is not the same as $\{2\}$ (the former is a set containing a set, whereas the latter is a set containing a number), $\{2\}$ is not an element of $\{\{\{2\}\}\}$.

8. a) true b) true c) false—see part (a) d) true
e) true—the one element in the set on the left is an element of the set on the right, and the sets are not equal
f) true—similar to part (e) g) false—the two sets are equal

10. The numbers 1, 3, 5, 7, and 9 form a subset of the set of all ten positive integers under discussion, as shown here.

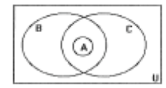


12. We put the subsets inside the supersets. Thus the answer is as shown.



14. We allow B and C to overlap, because we are told nothing about their relationship. The set A must be a subset of each of them, and that forces it to be positioned as shown. We cannot actually show the properness of the subset relationships in the diagram, because we don't know where the elements in B and C that are

not in A are located—there might be only one (which is in both B and C), or they might be located in portions of B and/or C outside the other. Thus the answer is as shown, but with the added condition that there must be at least one element of B not in A and one element of C not in A.



16. Since the empty set is a subset of every set, we just need to take a set B that contains $\emptyset$ as an element. Thus we can let $A = \emptyset$ and $B = \{\emptyset\}$ as the simplest example.

18. The cardinality of a set is the number of elements it has.
a) The empty set has no elements, so its cardinality is 0.
b) This set has one element (the empty set), so its cardinality is 1.
c) This set has two elements, so its cardinality is 2.
d) This set has three elements, so its cardinality is 3.

20. The union of all the sets in the power set of a set X must be exactly X. In other words, we can recover X from its power set, uniquely. Therefore the answer is yes.

22. a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.
b) This is the power set of $\{a\}$.
c) This set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set.
d) This is the power set of $\{a, b\}$.

24. By definition it is the set of all ordered pairs (c, p) such that c is a course and p is a professor.

26. We can conclude that $A = \emptyset$ or $B = \emptyset$. To prove this, suppose that neither A nor B were empty. Then there would be elements $a \in A$ and $b \in B$. This would give at last one element, namely (a, b), in $A \times B$, so $A \times B$ would not be the empty set. This contradiction shows that either A or B (or both, it goes without saying) is empty.

28. In each case the answer is a set of 3-tuples.
a) $\{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1),(b,y,0),(b,y,1),(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}$
b) $\{(0,x,a),(0,x,b),(0,x,c),(0,y,a),(0,y,b),(0,y,c),(1,x,a),(1,x,b),(1,x,c),(1,y,a),(1,y,b),(1,y,c)\}$
c) $\{(0,a,x),(0,a,y),(0,b,x),(0,b,y),(0,c,x),(0,c,y),(1,a,x),(1,a,y),(1,b,x),(1,b,y),(1,c,x),(1,c,y)\}$
d) $\{(x,x,x),(x,x,y),(x,y,x),(x,y,y),(y,x,x),(y,x,y),(y,y,x),(y,y,y)\}$

30. Suppose $A \ne B$ and neither A nor B is empty. We must prove that $A \times B \ne B \times A$. Since $A \ne B$, either we can find an element x that is in A but not B, or vice versa. The two cases are similar, so without loss of generality, let us assume that x is in A but not B. Also, since B is not empty, there is some element $y \in B$. Then (x, y) is in $A \times B$ by definition, but it is not in $B \times A$ since $x \notin B$. Therefore $A \times B \ne B \times A$.

32. The only difference between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ is parentheses, so for all practical purposes one can think of them as essentially the same thing. By Definition 9, the elements of $(A \times B) \times (C \times D)$ consist of ordered pairs (x, y), where $x \in A \times B$ and $y \in C \times D$, so the typical element of $(A \times B) \times (C \times D)$

looks like $((a,b),(c,d))$. By Definition 10, the elements of $A \times (B \times C) \times D$ consist of 3-tuples (a,x,d), where $a \in A$, $d \in D$, and $x \in B \times C$, so the typical element of $A \times (B \times C) \times D$ looks like $(a,(b,c),d)$. The structures $((a,b),(c,d))$ and $(a,(b,c),d)$ are different, even if they convey exactly the same information (the first is a pair, and the second is a 3-tuple). To be more precise, there is a natural one-to-one correspondence between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ given by $((a,b),(c,d)) \leftrightarrow (a,(b,c),d)$.

34. **a)** There is a real number whose cube is -1. This is true, since $x = -1$ is a solution.
b) There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is true—in fact, every integer satisfies this statement.
c) For every integer, the number obtained by subtracting 1 is again an integer. This is true.
d) The square of every integer is an integer. This is true.

36. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.
a) It is exactly the positive integers that satisfy this inequality. Therefore the truth set is $\{x \in \mathbf{Z} \mid x^3 \geq 1\} = \{x \in \mathbf{Z} \mid x \geq 1\} = \{1,2,3,\ldots\}$.
b) The square roots of 2 are not integers, so the truth set is the empty set, $\emptyset$.
c) Negative integers certainly satisfy this inequality, as do all positive integers greater than 1. However, $0 \not< 0^2$ and $1 \not< 1^2$. Thus the truth set is $\{x \in \mathbf{Z} \mid x < x^2\} = \{x \in \mathbf{Z} \mid x \neq 0 \wedge x \neq 1\} = \{\ldots,-3,-2,-1,2,3,\ldots\}$.

38. **a)** If $S \in S$, then by the defining condition for S we conclude that $S \notin S$, a contradiction.
b) If $S \notin S$, then by the defining condition for S we conclude that it is not the case that $S \notin S$ (otherwis[e] ...ment of S), again a contradiction.