looks like $((a,b),(c,d))$. By Definition 10, the elements of $A \times (B \times C) \times D$ consist of 3-tuples (a,x,d), where $a \in A$, $d \in D$, and $x \in B \times C$, so the typical element of $A \times (B \times C) \times D$ looks like $(a,(b,c),d)$. The structures $((a,b),(c,d))$ and $(a,(b,c),d)$ are different, even if they convey exactly the same information (the first is a pair, and the second is a 3-tuple). To be more precise, there is a natural one-to-one correspondence between $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ given by $((a,b),(c,d)) \leftrightarrow (a,(b,c),d)$.

34. a) There is a real number whose cube is -1. This is true, since $x = -1$ is a solution.
b) There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is true—in fact, every integer satisfies this statement.
c) For every integer, the number obtained by subtracting 1 is again an integer. This is true.
d) The square of every integer is an integer. This is true.

36. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.
a) It is exactly the positive integers that satisfy this inequality. Therefore the truth set is $\{x \in \mathbf{Z} \mid x^3 \geq 1\} = \{x \in \mathbf{Z} \mid x \geq 1\} = \{1,2,3,\ldots\}$.
b) The square roots of 2 are not integers, so the truth set is the empty set, $\emptyset$.
c) Negative integers certainly satisfy this inequality, as do all positive integers greater than 1. However, $0 \not< 0^2$ and $1 \not< 1^2$. Thus the truth set is $\{x \in \mathbf{Z} \mid x < x^2\} = \{x \in \mathbf{Z} \mid x \neq 0 \wedge x \neq 1\} = \{\ldots,-3,-2,-1,2,3,\ldots\}$.

38. a) If $S \in S$, then by the defining condition for S we conclude that $S \notin S$, a contradiction.
b) If $S \notin S$, then by the defining condition for S we conclude that it is not the case that $S \notin S$ (otherwise S would be an element of S), again a contradiction.

SECTION 2.2 Set Operations

2. a) $A \cap B$ b) $A \cap \overline{B}$, which is the same as $A - B$ c) $A \cup B$ d) $\overline{A} \cup \overline{B}$

4. Note that $A \subseteq B$.
a) $\{a,b,c,d,e,f,g,h\} = B$ b) $\{a,b,c,d,e\} = A$
c) There are no elements in A that are not in B, so the answer is $\emptyset$. d) $\{f,g,h\}$

6. a) $A \cup \emptyset = \{x \mid x \in A \vee x \in \emptyset\} = \{x \mid x \in A \vee \mathbf{F}\} = \{x \mid x \in A\} = A$
b) $A \cap U = \{x \mid x \in A \wedge x \in U\} = \{x \mid x \in A \wedge \mathbf{T}\} = \{x \mid x \in A\} = A$

8. a) $A \cup A = \{x \mid x \in A \vee x \in A\} = \{x \mid x \in A\} = A$
b) $A \cap A = \{x \mid x \in A \wedge x \in A\} = \{x \mid x \in A\} = A$

10. a) $A - \emptyset = \{x \mid x \in A \wedge x \notin \emptyset\} = \{x \mid x \in A \wedge \mathbf{T}\} = \{x \mid x \in A\} = A$
b) $\emptyset - A = \{x \mid x \in \emptyset \wedge x \notin A\} = \{x \mid \mathbf{F} \wedge x \notin A\} = \{x \mid \mathbf{F}\} = \emptyset$

12. We will show that these two sets are equal by showing that each is a subset of the other. Suppose $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$ by the definition of union. In the former case, we have $x \in A$, and in the latter case we have $x \in A$ and $x \in B$ by the definition of intersection; thus in any event, $x \in A$, so we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup (A \cap B)$ as well. Thus we have shown that the right-hand side is a subset of the left-hand side.

14. Since $A = (A - B) \cup (A \cap B)$, we conclude that $A = \{1,5,7,8\} \cup \{3,6,9\} = \{1,3,5,6,7,8,9\}$. Similarly $B = (B - A) \cup (A \cap B) = \{2,10\} \cup \{3,6,9\} = \{2,3,6,9,10\}$.

16. a) If x is in $A \cap B$, then perforce it is in A (by definition of intersection).
b) If x is in A, then perforce it is in $A \cup B$ (by definition of union).
c) If x is in $A - B$, then perforce it is in A (by definition of difference).
d) If $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \emptyset$.
e) The left-hand side consists precisely of those things that are either elements of A or else elements of B but not A, in other words, things that are elements of either A or B (or, of course, both). This is precisely the definition of the right-hand side.

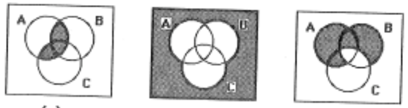
18. a) Suppose that $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In either case, certainly $x \in A \cup B \cup C$. This establishes the desired inclusion.
b) Suppose that $x \in A \cap B \cap C$. Then x is in all three of these sets. In particular, it is in both A and B and therefore in $A \cap B$, as desired.
c) Suppose that $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$ (we also know that $x \notin B$, but that won't be used here). Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.
d) To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no such x exists. So suppose that $x \in (A - C) \cap (C - B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies by definition that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.
e) To establish the equality, we need to prove inclusion in both directions. To prove that $(B - A) \cup (C - A) \subseteq (B \cup C) - A$, suppose that $x \in (B - A) \cup (C - A)$. Then either $x \in (B - A)$ or $x \in (C - A)$. Without loss of generality, assume the former (the proof in the latter case is exactly parallel.) Then $x \in B$ and $x \notin A$. From the first of these assertions, it follows that $x \in B \cup C$. Thus we can conclude that $x \in (B \cup C) - A$, as desired. For the converse, that is, to show that $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, suppose that $x \in (B \cup C) - A$. This means that $x \in (B \cup C)$ and $x \notin A$. The first of these assertions tells us that either $x \in B$ or $x \in C$. Thus either $x \in B - A$ or $x \in C - A$. In either case, $x \in (B - A) \cup (C - A)$. (An alternative proof could be given by using Venn diagrams, showing that both sides represent the same region.)

20. That $A \subseteq (A \cap B) \cup (A \cap \overline{B})$ follows from the fact that every element $x \in A$ is an element of either $A \cap B$ (if $x \in B$) or $A \cap \overline{B}$ (if $x \notin B$). On the other hand, if $x \in (A \cap B) \cup (A \cap \overline{B})$, then either $x \in A \cap B$ or $x \in A \cap \overline{B}$. In either case, $x \in A$ by the definition of intersection.

22. First we show that every element of the left-hand side must be in the right-hand side as well. If $x \in A \cap (B \cap C)$, then x must be in A and also in $B \cap C$. Hence x must be in A and also in B and in C. Since x is in both A and B, we conclude that $x \in A \cap B$. This, together with the fact that $x \in C$ tells us that $x \in (A \cap B) \cap C$, as desired. The argument in the other direction (if $x \in (A \cap B) \cap C$ then x must be in $A \cap (B \cap C)$) is nearly identical.

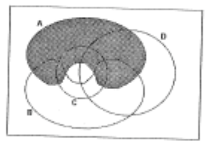
24. First suppose x is in the left-hand side. Then x must be in A but in neither B nor C. Thus $x \in A - C$, but $x \notin B - C$, so x is in the right-hand side. Next suppose that x is in the right-hand side. Thus x must be in $A - C$ and not in $B - C$. The first of these implies that $x \in A$ and $x \notin C$. But now it must also be the case that $x \notin B$, since otherwise we would have $x \in B - C$. Thus we have shown that x is in A but in neither B nor C, which implies that x is in the left-hand side.

26. The set is shaded in each case.

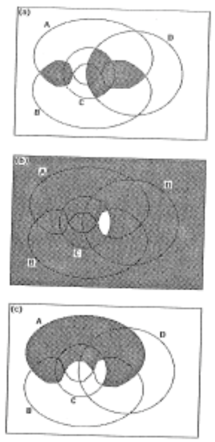


(a) (b) (c)

28. Here is a Venn diagram that can be used for four sets. Notice that sets A and B are not convex in this picture. We have shaded set A. Notice that each of the 16 different combinations are represented by a region.



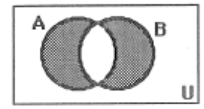
We can now shade in the appropriate regions for each of the expressions in this exercise.



30. a) We cannot conclude that $A = B$. For instance, if A and B are both subsets of C, then this equation will always hold, and A need not equal B.
b) We cannot conclude that $A = B$; let $C = \emptyset$, for example.
c) By putting the two conditions together, we *can* now conclude that $A = B$. By symmetry, it suffices to prove that $A \subseteq B$. Suppose that $x \in A$. There are two cases. If $x \in C$, then $x \in A \cap C = B \cap C$, which forces $x \in B$. On the other hand, if $x \notin C$, then because $x \in A \cup C = B \cup C$, we must have $x \in B$.

32. This is the set of elements in exactly one of these sets, namely $\{2, 5\}$.

34. The figure is as shown; we shade that portion of A that is not in B and that portion of B that is not in A.



36. There are precisely two ways that an item can be in either A or B but not both. It can be in A but not B (which is equivalent to saying that it is in $A - B$), or it can be in B but not A (which is equivalent to saying that it is in $B - A$). Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.

38. a) This is clear from the symmetry (between A and B) in the definition of symmetric difference.
b) We prove two things. To show that $A \subseteq (A \oplus B) \oplus B$, suppose $x \in A$. If $x \in B$, then $x \notin A \oplus B$, so x is an element of the right-hand side. On the other hand if $x \notin B$, then $x \in A \oplus B$, so again x is in the right-hand side. Conversely, suppose x is an element of the right-hand side. There are two cases. If $x \notin B$, then necessarily $x \in A \oplus B$, whence $x \in A$. If $x \in B$, then necessarily $x \notin A \oplus B$, and the only way for that to happen (since $x \in B$) is for x to be in A.

40. This is an identity; each side consists of those things that are in an odd number of the sets A, B, and C.

42. This is an identity; each side consists of those things that are in an odd number of the sets A, B, C, and D.

44. To count the elements of $A \cup B \cup C$ we proceed as follows. First we count the elements in each of the sets and add. This certainly gives us all the elements in the union, but we have overcounted. Each element in $A \cap B$, $A \cap C$, and $B \cap C$ has been counted twice. Therefore we subtract the cardinalities of these intersections to make up for the overcount. Finally, we have compensated a bit too much, since the elements of $A \cap B \cap C$ have now been counted three times and subtracted three times. We adjust by adding back the cardinality of $A \cap B \cap C$.

46. We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.
a) $A_n = \{\ldots, -2, -1, 0, 1, \ldots, n\}$ b) $A_1 = \{\ldots, -2, -1, 0, 1\}$

48. a) As i increases, the sets get smaller: $\cdots \subset A_3 \subset A_2 \subset A_1$. All the sets are subsets of A_1, which is the set of positive integers, $\mathbf{Z}^+$. It follows that $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z}^+$. Every positive integer is excluded from at least one of the sets (in fact from infinitely many), so $\bigcap_{i=1}^{\infty} A_i = \emptyset$.
b) All the sets are subsets of the set of natural numbers $\mathbf{N}$ (the nonnegative integers). The number 0 is in each of the sets, and every positive integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$ and $\bigcap_{i=1}^{\infty} A_i = \{0\}$.
c) As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of positive real numbers $\mathbf{R}^+$, and every positive real number is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}^+$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = (0, 1)$ (the interval of all real numbers between 0 and 1, exclusive).
d) This time, as in part (a), the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_1 = (1, \infty)$ (all real numbers greater than 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$.

50. **a)** 00 1110 0000 **b)** 10 1001 0001 **c)** 01 1100 1110

52. **a)** No elements are included, so this is the empty set.
b) All elements are included, so this is the universal set.

54. The bit string for the symmetric difference is obtained by taking the bitwise exclusive *OR* of the two bit strings for the two sets, since we want to include those elements that are in one set or the other but not both.

56. We can take the bitwise *OR* (for union) or *AND* (for intersection) of all the bit strings for these sets.

58. The successor set has one more element than the original set, namely the original set itself. Therefore the answer is $n+1$.

60. **a)** If the departments share the equipment, then the maximum number of each type is all that is required, so we want to take the union of the multisets, $A \cup B$.
b) Both departments will use the minimum number of each type, so we want to take the intersection of the multisets, $A \cap B$.
c) This will be the difference $B - A$ of the multisets.
d) If no sharing is allowed, then the university needs to purchase a quantity of each type of equipment that is the sum of the quantities used by the departments; this is the sum of the multisets, $A + B$.

62. Taking the maximum for each person, we have $S \cup T = \{0.6$ Alice, 0.9 Brian, 0.4 Fred