c) Since R and S each contain all the pairs (x,x), we know that $R\oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

d) Since R and S each contain all the pairs (x,x), we know that $R-S$ contains none of these pairs. Therefore $R-S$ is irreflexive.

e) Since R and S each contain all the pairs (x,x), so does $S \circ R$. Therefore $S \circ R$ is reflexive.

50. By definition, to say that R is antisymmetric is to say that $R \cap R^{-1}$ contains only pairs of the form (a,a). The statement we are asked to prove is just a rephrasing of this.

52. This is immediate from the definition, since R is reflexive if and only if it contains all the pairs (x,x), which in turn happens if and only if $\overline{R}$ contains none of these pairs, i.e., $\overline{R}$ is irreflexive.

54. We just apply the definition each time. We find that R^2 contains all the pairs in $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$ except $(2,3)$ and $(4,5)$; and R^3, R^4, and R^5 contain all the pairs.

56. We prove this by induction on n. There is nothing to prove in the basis step ($n=1$). Assume the inductive hypothesis that R^n is symmetric, and let $(a,c) \in R^{n+1} = R^n \circ R$. Then there is a $b \in A$ such that $(a,b) \in R$ and $(b,c) \in R^n$. Since R^n and R are symmetric, $(b,a) \in R$ and $(c,b) \in R^n$. Thus by definition $(c,a) \in R \circ R^n$. We will have completed the proof if we can show that $R \circ R^n = R^{n+1}$. This we do in two steps. First, composition of relations is associative, that is, $(R \circ S) \circ T = R \circ (S \circ T)$ for all relations with appropriate domains and codomains. (The proof of this is straightforward applications of the definition.) Second we show that $R \circ R^n = R^{n+1}$ by induction on n. Again the basis step is trivial. Under the inductive hypothesis, then, $R \circ R^{n+1} = R \circ (R^n \circ R) = (R \circ R^n) \circ R = R^{n+1} \circ R = R^{n+2}$, as desired.

SECTION 8.2 *n*-ary Relations and Their Applications

2. We have to find all the solutions to this equation, making sure to include all the permutations. The 4-tuples are $(6,1,1,1)$, $(1,6,1,1)$, $(1,1,6,1)$, $(1,1,1,6)$, $(3,2,1,1)$, $(3,1,2,1)$, $(3,1,1,2)$, $(2,3,1,1)$, $(2,1,3,1)$, $(2,1,1,3)$, $(1,3,2,1)$, $(1,3,1,2)$, $(1,2,3,1)$, $(1,2,1,3)$, $(1,1,3,2)$, and $(1,1,2,3)$.

4. Primary keys are the domains that have all different entries.

a) The only primary key is *Course*. b) The only primary key is *Course_number*.

c) The only primary key is *Course_number*. d) The only primary key is *Departure_time*.

6. We see that the *Professor* field by itself is not a key, since there is more than one 5-tuple containing the same professor. We can make the identification of the tuple unique by including the course number as well, or by including the time as well. Thus either *Professor–Course_number* or *Professor–Time* will work. Note, however, that either of these might not work if more data are added, since different departments can have the same course number, and a professor can be teaching two courses in the same room at the same time (e.g., a graduate course and the undergraduate version of that same course).

8. a) The ISBN is unique for each book, and it is probably the one and only primary key (and certainly the best one in any case).

b) This would work as long as there were not two books published the same year (date is usually given only as a year) with the same title. In practice, this could easily not happen.

c) This would work as long as there were not two books with the same title and the same number of pages. In practice, this could possibly not happen, although it is perhaps less likely than in part (b).

10. The selection operator picks out all the tuples that match the criteria. The 5-tuples in Table 7 that have A100 as their room are (Cruz, Zoology, 335, A100, 9:00 A.M.), (Cruz, Zoology, 412, A100, 8:00 A.M.), and (Farber, Psychology, 501, A100, 3:00 P.M.).

12. The selection operator picks out all the tuples that match the criteria. There is only one 4-tuple in Table 10 that has a quantity of at least 50 and project number 2, namely $(9191, 2, 80, 4)$.

14. We keep only the second, third, and fifth columns, obtaining (b, c, e).

16. The table uses columns 1, 2, and 4 of Table 8. We start by deleting columns 3 and 5 from Table 8. Since no rows are duplicates of earlier rows, this table is the answer.

Airline	*Flight_number*	*Destination*
Nadir	122	Detroit
Acme	221	Denver
Acme	122	Anchorage
Acme	323	Honolulu
Nadir	199	Detroit
Acme	222	Denver
Nadir	322	Detroit

18. By definition, there are $5 + 8 - 3 = 10$ components.

20. Both sides of this equation pick out the subset of R consisting of those n-tuples satisfying both conditions C_1 and C_2. This follows immediately from the definitions of conjunction and the selection operator.

22. Both sides of this equation pick out the set of n-tuples that satisfy condition C, and furthermore are in R or S (or both, of course). This follows immediately from the definitions of union and the selection operator.

24. Both sides of this equation pick out the set of n-tuples that satisfy condition C, and are in R and are not in S. This follows immediately from the definitions of set difference and the selection operator.

26. Note that we lose information when we delete columns. Therefore we might have more in the second set than in the first, since it could be easier to be in the intersection in the second case. A simple example would be to let $R = \{(a, b)\}$ and $S = \{(a, c)\}$, $n = 2$, $m = 1$, and $i_1 = 1$. Then $R \cap S = \emptyset$, so $P_1(R \cap S) = \emptyset$. On the other hand, $P_1(R) = P_1(S) = \{(a)\}$, so $P_1(R) \cap P_1(S) = \{(a)\}$.

28. This is similar to Example 13.

a) We apply the selection operator with the condition "$1000 \le$ Part_number ≤ 5000" to the 3-tuples given in Table 9, picking out those rows that have a part number in the indicated range. Then we choose the supplier field from those rows, and delete duplicates.

b) Five of the 3-tuples in the joined database satisfy the condition, namely $(23, 1092, 1)$, $(23, 1101, 3)$, $(31, 4975, 3)$, $(31, 3477, 2)$, and $(33, 1001, 1)$. The suppliers appearing here are $23, 31, 33$.

30. A primary key is a domain whose value determines the values of all the other domains. For this relation, this does not happen. The first domain is not a primary key, because, for example, the triples $(1, 2, 3)$ and $(1, 3, 5)$ are both in the relation (the terms form an arithmetic progression). Similarly, the triples $(1, 3, 5)$ and $(2, 3, 4)$ are both in the relation, so the second domain is not a key; and the triples $(1, 3, 5)$ and $(3, 4, 5)$ are both in the relation, so the third domain is not a key.

32. The primary key uniquely determines the n-tuple. Thus we can think of the n-tuple as a pair consisting of the primary key (in whichever field it lies) followed by the $(n-1)$-tuple consisting of the values from the other domains. The set of all such pairs is by definition the graph of the function from the subset of the domain of the primary key consisting of those values that appear, to the Cartesian product of the other $n-1$ domains.

SECTION 8.3 Representing Relations

2. In each case we use a 4×4 matrix, putting a 1 in position (i,j) if the pair (i,j) is in the relation and a 0 in position (i,j) if the pair (i,j) is not in the relation.

a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. a) Since the $(1,1)^{\text{th}}$ entry is a 1, $(1,1)$ is in the relation. Since $(1,3)^{\text{th}}$ entry is a 0, $(1,3)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,3)$, and $(4,4)$.

b) $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(3,3)$, $(3,4)$, $(4,1)$, and $(1,4)$

c) $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,4)$, $(4,1)$, and $(4,3)$

6. An asymmetric relation (see the preamble to Exercise 16 in Section 8.1) is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i), and there is no pair of 1's symmetrically placed around the main diagonal (i.e., we cannot have $m_{ij} = m_{ji} = 1$ for any values of i and j).

8. For reflexivity we want all 1's on the main diagonal; for irreflexivity we want all 0's on the main diagonal; for symmetry, we want the matrix to be symmetric about the main diagonal (equivalently, the matrix equals its transpose); for antisymmetry we want there never to be two 1's symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the main diagonal); and for transitivity we want the Boolean square of the matrix (the Boolean product of the matrix and itself) to be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.

a) Since some 1's and some 0's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,2)$ and $(2,1)$. Finally, the relation is not transitive; for example, the 1's in positions $(1,2)$ and $(2,3)$ would require a 1 in position $(1,3)$ if the relation were to be transitive.

b) Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. Since the matrix is not symmetric, the relation is not symmetric (look at positions $(1,2)$ and $(2,1)$, for example). The relation is antisymmetric since there are never two 1's symmetrically placed with respect to the main diagonal. Finally, the Boolean square of this matrix is not itself (look at position $(1,4)$ in the square), so the relation is not transitive.

c) Since there are all 0's on the main diagonal, this relation is not reflexive but is irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,2)$ and $(2,1)$, for example. Finally, the Boolean square of this matrix has a 1 in position $(1,1)$, so the relation is not transitive.

10. Note that the total number of entries in the matrix is $1000^2 = 1{,}000{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding 100, namely in position (a,b) where $a \le b$, as well as 1's along the diagonal. Thus the answer is the number of subsets of size 2 from a set of 100 elements, plus 1000, i.e., $C(1000,2) + 1000 = 499500 + 1000 = 500{,}500$.

b) There two 1's in each row of the matrix except the first and last rows, in which there is one 1. Therefore the answer is $998 \cdot 2 + 2 = 1998$.

c) There is a 1 in the matrix at each entry just above and to the left of the "anti-diagonal" (i.e., in positions $(1,999)$, $(2,998)$, $\ldots$, $(999,1)$. Therefore the answer is 999.

d) There is a 1 in the matrix at each entry on or above (to the left of) the "anti-diagonal." This is the same number of 1's as in part **(a)**, so the answer is again 500,500.

e) Every row has all 1's except for the first row, so the answer is $999 \cdot 1000 = 999{,}500$.

12. We take the transpose of the matrix, since we want the $(i,j)^{\text{th}}$ entry of the matrix for R^{-1} to be 1 if and only if the $(j,i)^{\text{th}}$ entry of R is 1.

14. a) The matrix for the union is formed by taking the join: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

b) The matrix for the intersection is formed by taking the meet: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

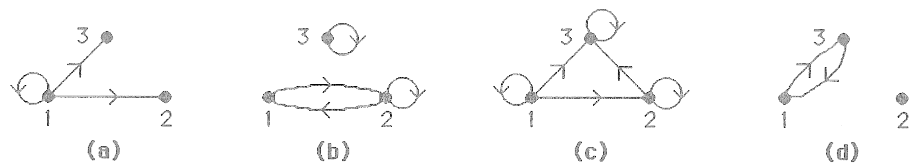
c) The matrix is the Boolean product $\mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

d) The matrix is the Boolean product $\mathbf{M}_{R_1} \odot \mathbf{M}_{R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

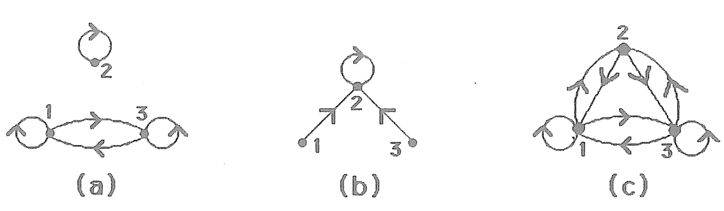
e) The matrix is the entrywise XOR: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

16. Since the matrix for R^{-1} is just the transpose of the matrix for R (see Exercise 12), the entries are the same collection of 0's and 1's, so there are k nonzero entries in $\mathbf{M}_{R^{-1}}$ as well.

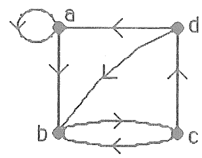
18. We draw the directed graphs, in each case with the vertex set being $\{1,2,3\}$ and an edge from i to j whenever (i,j) is in the relation.



20. In each case we draw a directed graph on three vertices with an edge from a to b for each pair (a,b) in the relation, i.e., whenever there is a 1 in position (a,b) in the matrix. In part **(a)**, for instance, we need an edge from 1 to itself since there is a 1 in position $(1,1)$ in the matrix, and an edge from 1 to 3, but no edge from 1 to 2.



22. We draw the directed graph with the vertex set being $\{a,b,c,d\}$ and an edge from i to j whenever (i,j) is in the relation.



24. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,c)\}$.

26. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,a),(c,c),(c,d),(d,d)\}$.

28. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$.

30. Clearly R is irreflexive if and only if there are no loops in the directed graph for R.

32. Recall that the relation is reflexive if there is a loop at each vertex; irreflexive if there are no loops at all; symmetric if edges appear only in **antiparallel** pairs (edges from one vertex to a second vertex and from the second back to the first); antisymmetric if there is no pair of antiparallel edges; asymmetric if is both antisymmetric and irreflexive; and transitive if all paths of length 2 (a pair of edges (x,y) and (y,z)) are accompanied by the corresponding path of length 1 (the edge (x,z)). The relation drawn in Exercise 26 is reflexive but not irreflexive since there are loops at each vertex. It is not symmetric, since, for instance, the edge (c,a) is present but not the edge (a,c). It is not antisymmetric, since both edges (a,b) and (b,a) are present. So it is not asymmetric either. It is not transitive, since the path $(c,a),(a,b)$ from c to b is not accompanied by the edge (c,b). The relation drawn in Exercise 27 is neither reflexive nor irreflexive since there are some loops but not a loop at each vertex. It is symmetric, since the edges appear in antiparallel pairs. It is not antisymmetric, since, for instance, both edges (a,b) and (b,a) are present. So it is not asymmetric either. It is not transitive, since edges (c,a) and (a,c) are present, but not (c,c). The relation drawn in Exercise 28 is reflexive and not irreflexive since there are loops at all vertices. It is symmetric but not antisymmetric or asymmetric. It is transitive; the only nontrivial paths of length 2 have the necessary loop shortcuts.

34. For each pair (a,b) of vertices (including the pairs (a,a) in which the two vertices are the same), if there is an edge from a to b, then erase it, and if there is no edge from a to b, put add it in.

36. We assume that the two relations are on the same set. For the union, we simply take the union of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in either of them. For intersection, we simply take the intersection of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there are edges from i to j in both of them. For symmetric difference, we simply take the symmetric difference of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in one, but not both, of them. Similarly, to form the difference, we take the difference of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in the first but not the second. To form the directed graph for the composition $S \circ R$ of relations R and S, we draw a directed graph on the same set of vertices and put in an edge from i to j whenever there is a vertex k such that there is an edge from i to k in R, and an edge from k to j in S.

34. All we need to do is make sure that all the pairs (x, x) are included. An easy way to accomplish this is to add them at the end, by setting $\mathbf{W} := \mathbf{W} \vee \mathbf{I}_n$.

SECTION 8.5 Equivalence Relations

2. a) This is an equivalence relation by Exercise 9 ($f(x)$ is x's age).
   b) This is an equivalence relation by Exercise 9 ($f(x)$ is x's parents).
   c) This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)
   d) This is not an equivalence relation since it is clearly not transitive.
   e) Again, just as in part (c), this is not transitive.

4. One relation is that a and b are related if they were born in the same U.S. state (with "not in a state of the U.S." counting as one state). Here the equivalence classes are the nonempty sets of students from each state. Another example is for a to be related to b if a and b have lived the same number of complete decades. The equivalence classes are the set of all 10-to-19 year-olds, the set of all 20-to-29 year-olds, and so on (the sets among these that are nonempty, that is). A third example is for a to be related to b if 10 is a divisor of the difference between a's age and b's age, where "age" means the whole number of years since birth, as of the first day of class. For each $i = 0, 1, \ldots, 9$, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit i.

6. One way to partition the classes would be by level. At many schools, classes have three-digit numbers, the first digit of which is approximately the level of the course, so that courses numbered 100–199 are taken by freshman, 200–299 by sophomores, and so on. Formally, two classes are related if their numbers have the same digit in the hundreds column; the equivalence classes are the set of all 100-level classes, the set of all 200-level classes, and so on. A second example would focus on department. Two classes are equivalent if they are offered by the same department; for example, MATH 154 is equivalent to MATH 372, but not to EGR 141. The equivalence classes are the sets of classes offered by each department (the set of math classes, the set of engineering classes, and so on). A third—and more egocentric—classification would be to have one equivalence class be the set of classes that you have completed successfully and the other equivalence class to be all the other classes. Formally, two classes are equivalent if they have the same answer to the question, "Have I completed this class successfully?"

8. Recall (Definition 4 in Section 2.4) that two sets have the same cardinality if there is a bijection (one-to-one and onto function) from one set to the other. We must show that R is reflexive, symmetric, and transitive. Every set has the same cardinality as itself because of the identity function. If f is a bijection from S to T, then f^{-1} is a bijection from T to S, so R is symmetric. Finally, if f is a bijection from S to T and g is a bijection from T to U, then $g \circ f$ is a bijection from T to U, so R is transitive (see Exercise 29 in Section 2.3).

The equivalence class of $\{1,2,3\}$ is the set of all three-element sets of real numbers, including such sets as $\{4,25,1948\}$ and $\{e,\pi,\sqrt{2}\}$. Similarly, $[\mathbf{Z}]$ is the set of all infinite countable sets of real numbers (see Section 2.4), such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set $\{1,2,3\}$ (it's too small) or the set of all real numbers (it's too big). See Section 2.4 for more on countable sets.

10. The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

12. This follows from Exercise 9, where f is the function that takes a bit string of length $n \geq 3$ to its last $n-3$ bits.

14. This follows from Exercise 9, where f is the function that takes a string of uppercase and lowercase English letters and changes all the lower case letters to their uppercase equivalents (and leaves the uppercase letters unchanged).

16. This follows from Exercise 9, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a,b) to a/b, since clearly $ad = bc$ if and only if $a/b = c/d$.

If we want an explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

18. a) This follows from Exercise 9, where the function f from the set of polynomials to the set of polynomials is the operator that takes the derivative n times—i.e., f of a function g is the function $g^{(n)}$. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "n^{th} derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."

b) The third derivative of x^4 is $24x$. Since the third derivative of a polynomial of degree 2 or less is 0, the polynomials of the form $x^4 + ax^2 + bx + c$ have the same third derivative. Thus these are the functions in the same equivalence class as f.

20. This follows from Exercise 9, where the function f from the set of people to the set of Web-traversing behaviors starting at the given particular Web page takes the person to the behavior that person exhibited.

22. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is an equivalence relation, satisfying all three properties. The equivalence classes are $\{a,d\}$ and $\{b,c\}$.

24. a) This is not an equivalence relation, since it is not symmetric.

b) This is an equivalence relation; one equivalence class consists of the first and third elements, and the other consists of the second and fourth elements.

c) This is an equivalence relation; one equivalence class consists of the first, second, and third elements, and the other consists of the fourth element.

26. Only part (a) and part (c) are equivalence relations. In part (a) each element is in an equivalence class by itself. In part (c) the elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

28. Only part (a) and part (d) are equivalence relations. In part (a) there is one equivalence class for each $n \in \mathbf{Z}$, and it contains all those functions whose value at 1 is n. In part (d) there really is no good way to describe the equivalence classes. For one thing, the set of equivalence classes is uncountable. For each function $f : \mathbf{Z} \to \mathbf{Z}$, there is the equivalence class consisting of all those functions g for which there is a constant C such that $g(n) = f(n) + C$ for all $n \in \mathbf{Z}$.

30. **a)** all the strings whose first three bits are 010 **b)** all the strings whose first three bits are 101
c) all the strings whose first three bits are 111 **d)** all the strings whose first three bits are 010

32. Since two strings are related if they agree beyond their first 3 bits, the equivalence class of a bit string $xyzt$, where x, y, and z are bits, and t is a bit string, is the set of all bit strings of the form $x'y'z't$, where x', y', and z' are any bits.
a) the set of all bit strings of length 3 (take $t = \lambda$ in the formulation given above)
b) the set of all bit strings of length 4 that end with a 1
c) the set of all bit strings of length 5 that end 11
d) the set of all bit strings of length 8 that end 10101

34. **a)** Since this string has length less than 5, its equivalence class consists only of itself.
b) This is similar to part (a): $[1011]_{R_5} = \{1011\}$.
c) Since this string has length 5, its equivalence class consists of all strings that start 11111.
d) This is similar to part (c): $[01010101]_{R_5} = \{01010s \mid s \text{ is any bit string}\}$.

36. In each case, the equivalence class of 4 is the set of all integers congruent to 4, modulo m.
a) $\{4 + 2n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 0, 2, 4, \ldots\}$ **b)** $\{4 + 3n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 1, 4, 7, \ldots\}$
c) $\{4 + 6n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 4, 10, 16, \ldots\}$ **d)** $\{4 + 8n \mid n \in \mathbf{Z}\} = \{\ldots, -4, 4, 12, 20, \ldots\}$

38. In each case we need to allow all strings that agree with the given string if we ignore the case in which the letters occur.
a) $\{NO, No, nO, no\}$
b) $\{YES, YEs, YeS, Yes, yES, yEs, yeS, yes\}$
c) $\{HELP, HELp, HElP, HElp, HeLP, HeLp, HelP, Help, hELP, hELp, hElP, hElp, heLP, heLp, helP, help\}$

40. **a)** By our observation in the solution to Exercise 16, the equivalence class of $(1, 2)$ is the set of all pairs (a, b) such that the fraction a/b equals $1/2$.
b) Again by our observation, the equivalence classes are the positive rational numbers. (Indeed, this is the way one can rigorously define what a rational number is, and this is why fractions are so difficult for children to understand.)

42. **a)** This is a partition, since it satisfies the definition.
b) This is not a partition, since the subsets are not disjoint.
c) This is a partition, since it satisfies the definition.
d) This is not a partition, since the union of the subsets leaves out 0.

44. **a)** This is clearly a partition. **b)** This is not a partition, since 0 is in neither set.
c) This is a partition by the division algorithm.
d) This is a partition, since the second set mentioned is the set of all number between -100 and 100, inclusive.
e) The first two sets are not disjoint (4 is in both), so this is not a partition.

46. a) This is a partition, since it satisfies the definition.

b) This is a partition, since it satisfies the definition.

c) This is not a partition, since the intervals are not disjoint (they share endpoints).

d) This is not a partition, since the union of the subsets leaves out the integers.

e) This is a partition, since it satisfies the definition.

f) This is a partition, since it satisfies the definition. Each equivalence class consists of all real numbers with a fixed fractional part.

48. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

b) $\{(a,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(f,e),(f,f),(g,g)\}$

c) $\{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d),$ $(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

d) $\{(a,a),(a,c),(a,e),(a,g),(c,a),(c,c),(c,e),(c,g),(e,a),(e,c),(e,e),(e,g),(g,a),(g,c),(g,e),(g,g),$ $(b,b),(b,d),(d,b),(d,d),(f,f)\}$

50. We need to show that every equivalence class consisting of people living in the same county (or parish) and same state is contained in an equivalence class of all people living in the same state. This is clear. The equivalence class of all people living in county c in state s is a subset of the set of people living in state s.

52. We are asked to show that every equivalence class for R_4 is a subset of some equivalence class for R_3. Let $[y]_{R_4}$ be an arbitrary equivalence class for R_4. We claim that $[y]_{R_4} \subseteq [y]_{R_3}$; proving this claim finishes the proof. To show that one set is a subset of another set, we choose an arbitrary bit string x in the first set and show that it is also an element of the second set. In this case since $y \in [x]_{R_4}$, we know that y is equivalent to x under R_4, that is, that either $y = x$ or y and x are each at least 4 bits long and agree on their first 4 bits. Because strings that are at least 4 bits long and agree on their first 4 bits perforce are at least 3 bits long and agree on their first 3 bits, we know that either $y = x$ or y and x are each at least 3 bits long and agree on their first 3 bits. This means that y is equivalent to x under R_3, that is, that $y \in [x]_{R_3}$.

54. First, suppose that $R_1 \subseteq R_2$. We must show that P_1 is a refinement of P_2. Let $[a]_{R_1}$ be an equivalence class in P_1. We must show that $[a]_{R_1}$ is contained in an equivalence class in P_2. In fact, we will show that $[a]_{R_1} \subseteq [a]_{R_2}$. To this end, let $b \in [a]_{R_1}$. Then $(a,b) \in R_1 \subseteq R_2$. Therefore $b \in [a]_{R_2}$, as desired.

Conversely, suppose that P_1 is a refinement of P_2. Since $a \in [a]_{R_2}$, the definition of "refinement" forces $[a]_{R_1} \subseteq [a]_{R_2}$ for all $a \in A$. This means that for all $b \in A$ we have $(a,b) \in R_1 \to (a,b) \in R_2$; in other words, $R_1 \subseteq R_2$.

56. a) This need not be an equivalence relation, since it need not be transitive.

b) Since the intersection of reflexive, symmetric, and transitive relations also have these properties (see Section 8.1), the intersection of equivalence relations is an equivalence relation.

c) This will never be an equivalence relation on a nonempty set, since it is not reflexive.

58. This exercise is very similar to Exercise 59, and the reader should look at the solution there for details.

a) As in Exercise 59, the motions of the bracelet form a dihedral group, in this case consisting of six motions: rotations of 0°, 120°, and 240°, and three reflections, each keeping one bead fixed and interchanging the other two. The composition of any two of these operations is again one of these operations. The 0° rotation plays the role of the identity, which says that the relation is reflexive. Each operation has an inverse (reflections are their own inverses, the 0° rotation is its own inverse, and the 120° and 240° rotations are inverses of each other); this proves symmetry. And transitivity follows from the group table.

b) The equivalence classes are the indistinguishable bracelets. If we denote a bracelet by the colors of its beads, then these classes can be described as RRR, WWW, BBB, RRW, RRB, WWR, WWB, BBR, BBW, and RWB. Note that once we specify the colors, then every two bracelets with those colors are equivalent. This would not be the case if there were four or more beads, however. For example, in a 4-bead bracelet with two reds and two whites, the bracelet in which the red beads are adjacent is not equivalent to the one in which they are not.

60. a) In Exercise 25 of Section 3.2, we showed that $f(x)$ is $\Theta(g(x))$ if and only if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. To show that R is reflexive, we need to show that $f(x)$ is $O(f(x))$, which is clear by taking $C = 1$ and $k = 1$ in the definition. Symmetry is immediate from the definition, since if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$, then $g(x)$ is $O(f(x))$ and $f(x)$ is $O(g(x))$. Finally, transitivity follows immediately from the transitive of the "is big-O of" relation, which was proved in Exercise 17 of Section 3.2.

b) This is the class of all functions that asymptotically (i.e., as $n \to \infty$) grow just as fast as a multiple of $f(n) = n^2$. So, for example, functions such as $g(n) = 5n^2 + \log n$, or $g(n) = (n^3 - 17)/(100n + 10^{10})$ belong to this class, but $g(n) = n^{2.01}$ does not (it grows too fast), and $g(n) = n^2/\log n$ does not (it grows too slowly). Another way to express this class is to say that it is the set of all functions g such that there exist constants positive C_1 and C_2 such that the ratio $f(n)/g(n)$ always lies between C_1 and C_2.

62. We will count partitions instead, since equivalence relations are in one-to-one correspondence with partitions. Without loss of generality let the set be $\{1, 2, 3, 4\}$. There is 1 partition in which all the elements are in the same set, namely $\{\{1, 2, 3, 4\}\}$. There are 4 partitions in which the sizes of the sets are 1 and 3, namely $\{\{1\}, \{2, 3, 4\}\}$ and three more like it. There are 3 partitions in which the sizes of the sets are 2 and 2, namely $\{\{1, 2\}, \{3, 4\}\}$ and two more like it. There are 6 partitions in which the sizes of the sets are 2, 1, and 1, namely $\{\{1, 2\}, \{3\}, \{4\}\}$ and five more like it. Finally, there is 1 partition in which all the elements are in separate sets. This gives a total of 15. To actually list the 15 relations would be tedious.

64. No. Here is a counterexample. Start with $\{(1, 2), (3, 2)\}$ on the set $\{1, 2, 3\}$. Its transitive closure is itself. The reflexive closure of that is $\{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}$. The symmetric closure of that is $\{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$. The result is not transitive; for example, $(1, 3)$ is missing. Therefore this is not an equivalence relation.

66. We end up with the original partition P.

68. We will develop this recurrence relation in the context of partitions of the set $\{1, 2, \ldots, n\}$. Note that $p(0) = 1$, since there is only one way to partition the empty set (namely, into the empty collection of subsets). For warm-up, we also note that $p(1) = 1$, since $\{\{1\}\}$ is the only partition of $\{1\}$; that $p(2) = 2$, since we can partition $\{1, 2\}$ either as $\{\{1, 2\}\}$ or as $\{\{1\}, \{2\}\}$; and that $p(3) = 5$, since there are the following partitions: $\{\{1, 2, 3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$, $\{\{1\}, \{2\}, \{3\}\}$. Now to partition $\{1, 2, \ldots, n\}$, we first decide how many other elements of this set will go into the same subset as n goes into. Call this number j, and note that j can take any value from 0 through $n - 1$. Once we have determined j, we can specify the partition by deciding on the subset of j elements from $\{1, 2, \ldots, n - 1\}$ that will go into the same subset as n (and this can be done in $C(n - 1, j)$ ways), and then we need to decide how to partition the remaining $n - 1 - j$ elements (and this can be done in $p(n - j - 1)$ ways). The given recurrence relation now follows.

SECTION 8.6 Partial Orderings

2. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) This relation is not reflexive because 1 is not related to itself. Therefore R is not a partial ordering. The relation is antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c \neq 1$ so a is related to c.

b) This is a partial ordering, because it is reflexive and the pairs $(2,0)$ and $(2,3)$ will not introduce any violations of antisymmetry or transitivity.

c) This is not a partial ordering, because it is not transitive: $3\,R\,1$ and $1\,R\,2$, but 3 is not related to 2. It is reflexive and the pairs $(1,2)$ and $(3,1)$ will not introduce any violations of antisymmetry.

d) This is not a partial ordering, because it is not transitive: $1\,R\,2$ and $2\,R\,0$, but 1 is not related to 0. It is reflexive and the nonreflexive pairs will not introduce any violations of antisymmetry.

e) The relation is clearly reflexive, but it is not antisymmetric ($0\,R\,1$ and $1\,R\,0$, but $0 \neq 1$) and not transitive ($2\,R\,0$ and $0\,R\,1$, but 2 is not related to 1).

4. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since there surely are unequal people of the same height (to whatever degree of precision heights are measured), this relation is not antisymmetric, so (S,R) cannot be a poset.

b) Since nobody weighs more than herself, this relation is not reflexive, so (S,R) cannot be a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is a descendant of b, then b cannot be a descendant of a (for one thing, a descendant needs to be born after any ancestor), so the relation is vacuously antisymmetric. If a is a descendant of b, and b is a descendant of c, then by the way "descendant" is defined, we know that a is a descendant of c; thus R is transitive.

d) This relation is not reflexive, because anyone and himself have a common friend.

6. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial order. (It is the smallest partial order; reflexivity insures that every partial order contains at least all the pairs (a,a).)

b) This is not a poset, since the relation is not reflexive, although it is antisymmetric and transitive. Any relation of this sort can be turned into a partial ordering by adding in all the pairs (a,a).

c) This is a poset, very similar to Example 1.

d) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

8. a) This relation is $\{(1,1),(1,3),(2,1),(2,2),(3,3)\}$. It is clearly reflexive and antisymmetric. The only pairs that might present problems with transitivity are the nondiagonal pairs, $(2,1)$ and $(1,3)$. If the relation were to be transitive, then we would also need the pair $(2,3)$ in the relation. Since it is not there, the relation is not a partial order.

b) Reasoning as in part (a), we see that this relation is a partial order, since the pair $(3,1)$ can cause no problem with transitivity.

c) A little trial and error shows that this relation is not transitive ($(1,3)$ and $(3,4)$ are present, but not $(1,4)$) and therefore not a partial order.

This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

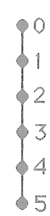
12. This follows immediately from the definition. Clearly R^{-1} is reflexive if R is. For antisymmetry, suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, whence $(b,a) \notin R^{-1}$. Finally, if $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, so $(c,a) \in R$ (since R is transitive), and therefore $(a,c) \in R^{-1}$; thus R^{-1} is transitive.

14. **a)** These are comparable, since $5 \mid 15$.
b) These are not comparable since neither divides the other.
c) These are comparable, since $8 \mid 16$.
d) These are comparable, since $7 \mid 7$.

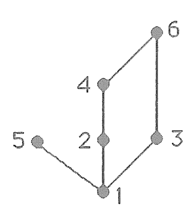
16. **a)** We need either a number less than 2 in the first coordinate, or a 2 in the first coordinate and a number less than 3 in the second coordinate. Therefore the answer is $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, and $(2,2)$.
b) We need either a number greater than 3 in the first coordinate, or a 3 in the first coordinate and a number greater than 1 in the second coordinate. Therefore the answer is $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(3,2)$, $(3,3)$, and $(3,4)$.
c) The Hasse diagram is a straight line with 16 points on it, since this is a total order. The pair $(4,4)$ is at the top, $(4,3)$ beneath it, $(4,2)$ beneath that, and so on, with $(1,1)$ at the bottom. To save space, we will not actually draw this picture.

18. **a)** The string *quack* comes first, since it is an initial substring of *quacking*, which comes next (since the other three strings all begin *qui*, not *qua*). Similarly, these last three strings are in the order *quick*, *quicksand*, *quicksilver*.
b) The order is *open*, *opened*, *opener*, *opera*, *operand*.
c) The order is *zero*, *zoo*, *zoological*, *zoology*, *zoom*.

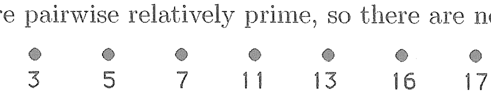
20. The Hasse diagram for this total order is a straight line, as shown, with 0 at the top (it is the "largest" element under the "is greater than or equal to" relation) and 5 at the bottom.



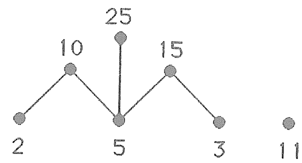
22. In each case we put a above b and draw a line between them if $b \mid a$ but there is no element c other than a and b such that $b \mid c$ and $c \mid a$.
a) Note that 1 divides all numbers, so the numbers on the second level from the bottom are the primes.



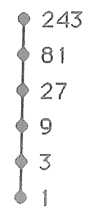
b) In this case these numbers are pairwise relatively prime, so there are no lines in the Hasse diagram.



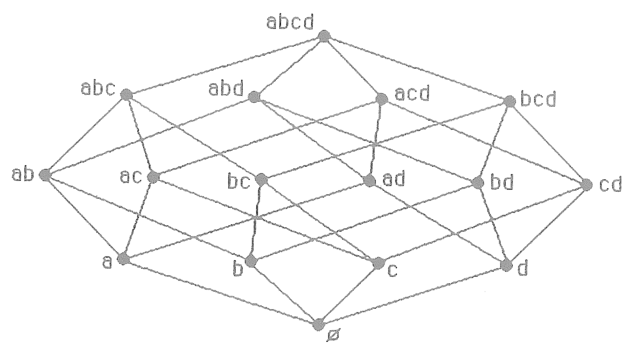
c) Note that we can place the points as we wish, as long as a is above b when $b \mid a$.



d) In this case these numbers each divide the next, so the Hasse diagram is a straight line.



24. This picture is a four-dimensional cube. We draw the sets with k elements at level k: the empty set at level 0 (the bottom), the entire set at level 4 (the top).



26. The procedure is the same as in Exercise 25: $\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,d),(d,d),(e,e)\}$

28. In this problem $a \preceq b$ when $a \mid b$. For (a,b) to be in the covering relation, we need a to be a proper divisor of b but we also must have no element in our set $\{1, 2, 3, 4, 6, 12\}$ being a proper multiple of a and a proper divisor of b. For example, $(2,12)$ is not in the covering relation, since $2 \mid 6$ and $6 \mid 12$. With this understanding it is easy to list the pairs in the covering relation: $(1,2)$, $(1,3)$, $(2,4)$, $(2,6)$, $(3,6)$, $(4,12)$, and $(6,12)$.

30. In the Hasse diagram, if x is lower than y and there is an edge joining x and y, then $x \prec y$. We claim that in this case (x,y) is also in the covering relation. Indeed, the definition of the Hasse diagram stated that there is to be no edge from x to y if there is some z with $x \prec z \prec y$ (such edges forced by transitivity were to be removed in the construction). This is equivalent to having (x,y) in the covering relation. Conversely, if (x,y) is in the covering relation, the $x \prec y$, and there is no z that would have caused the edge between x and y to be removed in the construction of the Hasse diagram. This completes the proof.

32. a) The maximal elements are the ones with no other elements above them, namely l and m.

b) The minimal elements are the ones with no other elements below them, namely a, b, and c.

c) There is no greatest element, since neither l nor m is greater than the other.

d) There is no least element, since neither a nor b is less than the other.

e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l, and m are the elements fitting this description.

f) Since k is less than both l and m, it is the least upper bound of a, b, and c.

g) No element is less than both f and h, so there are no lower bounds.

h) Since there are no lower bounds, there can be no greatest lower bound.

34. The reader should draw the Hasse diagram to aid in answering these questions.

a) Clearly the numbers 27, 48, 60, and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72, however, so they are not maximal.

b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9.

c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.

d) There is no least element, since there is no number in the set that divides both 2 and 9.

e) We need to find numbers in the list that are multiples of both 2 and 9. Clearly 18, 36, and 72 are the numbers we are looking for.

f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.

g) We need to find numbers in the list that are divisors of both 60 and 72. Clearly 2, 4, 6, and 12 are the numbers we are looking for.

h) Of the numbers we found in the previous part, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

36. a) One example is the natural numbers under "is less than or equal to." Here 1 is the (only) minimal element, and there are no maximal elements.

b) Dual to part (a), the answer is the natural numbers under "is greater than or equal to."

c) Combining the answers for the first two parts, we look at the set of integers under "is less than or equal to." Clearly there are no maximal or minimal elements.

38. Reflexivity is clear from the definition. To show antisymmetry, suppose that $a_1 \ldots a_m < b_1 \ldots b_n$, and let $t = \min(m,n)$. This means that either $a_1 \ldots a_t = b_1 \ldots b_t$ and $m < n$, so that $b_1 \ldots b_n \not< a_1 \ldots a_m$, or else $a_1 \ldots a_t < b_1 \ldots b_t$, so that $b_1 \ldots b_t \not< a_1 \ldots a_t$ and hence again $b_1 \ldots b_n \not< a_1 \ldots a_m$. Finally for transitivity, suppose that $a_1 \ldots a_m < b_1 \ldots b_n < c_1 \ldots c_p$. Let $t = \min(m,n)$, $r = \min(n,p)$, $s = \min(m,p)$, and $l = \min(m,n,p)$. Now if $a_1 \ldots a_l < b_1 \ldots b_l < c_1 \ldots c_l$, then clearly $a_1 \ldots a_m < c_1 \ldots c_p$. Otherwise, without loss of generality we may assume that $a_1 \ldots a_l = b_1 \ldots b_l$. If $l = t$, then $m < n$ and $m \le p$. Furthermore, either $b_1 \ldots b_r < c_1 \ldots c_r$, or $b_1 \ldots b_r = c_1 \ldots c_r$ and $n < p$. In the former case, if $r > l$, then since $p > m$ we have $a_1 \ldots a_m < c_1 \ldots c_p$, whereas if $r = l$, then $a_1 \ldots a_l < c_1 \ldots c_l$. In the latter case, $a_1 \ldots a_s = c_1 \ldots c_s$ and $m < p$, so again $a_1 \ldots a_m < c_1 \ldots c_p$. If $l < t$, then we must have $b_1 \ldots b_l < c_1 \ldots c_l$, whence $a_1 \ldots a_l < c_1 \ldots c_l$.

40. a) If x and y are both greatest elements, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

b) This is dual to part (a). If x and y are both least elements, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

42. a) If x and y are both least upper bounds, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

b) This is dual to part (a). If x and y are both greatest lower bounds, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

44. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least per bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the ıaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the o numbers.

This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets and B is $A \cup B$, and their l.u.b. is $A \cap B$.

the duality in the definitions, the greatest lower bound of two elements of S under R is their least upper und under R^{-1}, and their least upper bound under R is their greatest lower bound under R^{-1}. Therefore, (S, R) is a lattice (i.e., all the l.u.b.'s and g.l.b.'s exist), then so is (S, R^{-1}).

e need to verify the various defining properties of a lattice. First, we need to show that S is a poset under e given $\preceq$ relation. Clearly $(A, C) \preceq (A, C)$, since $A \leq A$ and $C \subseteq C$; thus we have established reflexivity. r antisymmetry, suppose that $(A_1, C_1) \preceq (A_2, C_2)$ and $(A_2, C_2) \preceq (A_1, C_1)$. This means that $A_1 \leq A_2$, $\subseteq C_2$, $A_2 \leq A_1$, and $C_2 \subseteq C_1$. By the properties of $\leq$ and $\subseteq$ it immediately follows that $A_1 = A_2$ ıd $C_1 = C_2$, so $(A_1, C_1) = (A_2, C_2)$. Transitivity is proved in a similar way, using the transitivity of $\leq$ ıd $\subseteq$. Second, we need to show that greatest lower bounds and least upper bounds exist. Suppose that $_1, C_1)$ and (A_2, C_2) are two elements of S; we claim that $(\min(A_1, A_2), C_1 \cap C_2)$ is their greatest lower ound. Clearly $\min(A_1, A_2) \leq A_1$ and $\min(A_1, A_2) \leq A_2$; and $C_1 \cap C_2 \subseteq C_1$ and $C_1 \cap C_2 \subseteq C_2$. Therefore $\text{in}(A_1, A_2), C_1 \cap C_2) \preceq (A_1, C_1)$ and $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_2, C_2)$, so this is a lower bound. On the her hand, if (A, C) is any lower bound, then $A \leq A_1$, $A \leq A_2$, $C \subseteq C_1$, and $C \subseteq C_2$. It follows from the operties of $\leq$ and $\subseteq$ that $A \leq \min(A_1, A_2)$ and $C \subseteq C_1 \cap C_2$. Therefore $(A, C) \preceq (\min(A_1, A_2), C_1 \cap C_2)$. his means that $(\min(A_1, A_2), C_1 \cap C_2)$ is the greatest lower bound. The proof that $(\max(A_1, A_2), C_1 \cup C_2)$ the least upper bound is exactly dual to this argument.

his issue was already dealt with in our solution to Exercise 44, parts (b) and (c). If $(S, \leq)$ is a total (linear) der, then the least upper bound of two elements is the larger one, and their greatest lower bound is the naller.

y Exercise 50, we can try to choose our examples from among total orders, such as subsets of $\mathbf{Z}$ under $\leq$.
) $(\mathbf{Z}, \leq)$ b) $(\mathbf{Z}^+, \leq)$ c) $(\mathbf{Z}^-, \leq)$, where $\mathbf{Z}^-$ is the set of negative integers d) $(\{1\}, \leq)$

each case, the issue is whether every subset contains a least element.

) The is well-ordered, since the minimum element in each set is its smallest element.

) This is not well-ordered. For example, the set $\{\frac{1}{n} \mid n \in \mathbf{N}\}$ contains no minimum element.

) This is a finite totally-ordered set, so it is well-ordered.

) This is well-ordered, since has the same structure as the positive integers under $\leq$, because $x \geq y$ if and ıly if $-x \leq -y$.

et x_0 and x_1 be two elements in the dense poset, with $x_0 \prec x_1$ (guaranteed by the conditions stated). By ensity, there is an element x_2 between x_0 and x_1, i.e., with $x_0 \prec x_2 \prec x_1$. Again by density, there is an ement x_3 between x_0 and x_2, i.e., with $x_0 \prec x_3 \prec x_2$. We continue in this manner and have produced an finite decreasing sequence: $\cdots \prec x_4 \prec x_3 \prec x_2 \prec x_1$. Thus the poset is not well-founded.

is not well-founded because of the infinite decreasing sequence $\cdots \prec aaab \prec aab \prec ab \prec b$. It is not dense, ecause there is no element between a and aa in this order.

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the two numbers.

d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

46. By the duality in the definitions, the greatest lower bound of two elements of S under R is their least upper bound under R^{-1}, and their least upper bound under R is their greatest lower bound under R^{-1}. Therefore, if (S, R) is a lattice (i.e., all the l.u.b.'s and g.l.b.'s exist), then so is (S, R^{-1}).

48. We need to verify the various defining properties of a lattice. First, we need to show that S is a poset under the given $\preceq$ relation. Clearly $(A, C) \preceq (A, C)$, since $A \leq A$ and $C \subseteq C$; thus we have established reflexivity. For antisymmetry, suppose that $(A_1, C_1) \preceq (A_2, C_2)$ and $(A_2, C_2) \preceq (A_1, C_1)$. This means that $A_1 \leq A_2$, $C_1 \subseteq C_2$, $A_2 \leq A_1$, and $C_2 \subseteq C_1$. By the properties of $\leq$ and $\subseteq$ it immediately follows that $A_1 = A_2$ and $C_1 = C_2$, so $(A_1, C_1) = (A_2, C_2)$. Transitivity is proved in a similar way, using the transitivity of $\leq$ and $\subseteq$. Second, we need to show that greatest lower bounds and least upper bounds exist. Suppose that (A_1, C_1) and (A_2, C_2) are two elements of S; we claim that $(\min(A_1, A_2), C_1 \cap C_2)$ is their greatest lower bound. Clearly $\min(A_1, A_2) \leq A_1$ and $\min(A_1, A_2) \leq A_2$; and $C_1 \cap C_2 \subseteq C_1$ and $C_1 \cap C_2 \subseteq C_2$. Therefore $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_1, C_1)$ and $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_2, C_2)$, so this is a lower bound. On the other hand, if (A, C) is any lower bound, then $A \leq A_1$, $A \leq A_2$, $C \subseteq C_1$, and $C \subseteq C_2$. It follows from the properties of $\leq$ and $\subseteq$ that $A \leq \min(A_1, A_2)$ and $C \subseteq C_1 \cap C_2$. Therefore $(A, C) \preceq (\min(A_1, A_2), C_1 \cap C_2)$. This means that $(\min(A_1, A_2), C_1 \cap C_2)$ is the greatest lower bound. The proof that $(\max(A_1, A_2), C_1 \cup C_2)$ is the least upper bound is exactly dual to this argument.

50. This issue was already dealt with in our solution to Exercise 44, parts (b) and (c). If $(S, \leq)$ is a total (linear) order, then the least upper bound of two elements is the larger one, and their greatest lower bound is the smaller.

52. By Exercise 50, we can try to choose our examples from among total orders, such as subsets of $\mathbf{Z}$ under $\leq$.
a) $(\mathbf{Z}, \leq)$ b) $(\mathbf{Z}^+, \leq)$ c) $(\mathbf{Z}^-, \leq)$, where $\mathbf{Z}^-$ is the set of negative integers d) $(\{1\}, \leq)$

54. In each case, the issue is whether every subset contains a least element.

a) The is well-ordered, since the minimum element in each set is its smallest element.

b) This is not well-ordered. For example, the set $\{\frac{1}{n} \mid n \in \mathbf{N}\}$ contains no minimum element.

c) This is a finite totally-ordered set, so it is well-ordered.

d) This is well-ordered, since has the same structure as the positive integers under $\leq$, because $x \geq y$ if and only if $-x \leq -y$.

56. Let x_0 and x_1 be two elements in the dense poset, with $x_0 \prec x_1$ (guaranteed by the conditions stated). By density, there is an element x_2 between x_0 and x_1, i.e., with $x_0 \prec x_2 \prec x_1$. Again by density, there is an element x_3 between x_0 and x_2, i.e., with $x_0 \prec x_3 \prec x_2$. We continue in this manner and have produced an infinite decreasing sequence: $\cdots \prec x_4 \prec x_3 \prec x_2 \prec x_1$. Thus the poset is not well-founded.

58. It is not well-founded because of the infinite decreasing sequence $\cdots \prec aaab \prec aab \prec ab \prec b$. It is not dense, because there is no element between a and aa in this order.