CHAPTER 9
Graphs

SECTION 9.1 Graphs and Graph Models

2. a) A simple graph would be the model here, since there are no parallel edges or loops, and the edges are undirected.

b) A multigraph would, in theory, be needed here, since there may be more than one interstate highway between the same pair of cities.

c) A pseudograph is needed here, to allow for loops.

4. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

6. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

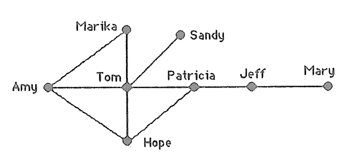
8. This is a directed multigraph; the edges are directed, and there are parallel edges.

10. The graph in Exercise 3 is simple. The multigraph in Exercise 4 can be made simple by removing one of the edges between a and b, and two of the edges between b and d. The pseudograph in Exercise 5 can be made simple by removing the three loops and one edge in each of the three pairs of parallel edges. The multigraph in Exercise 6 can be made simple by removing one of the edges between a and c, and one of the edges between b and d. The other three are not undirected graphs. (Of course removing any supersets of the answers given here are equally valid answers; in particular, we could remove *all* the edges in each case.)

12. If $u\,R\,v$, then there is an edge joining vertices u and v, and since the graph is undirected, this is also an edge joining vertices v and u. This means that $v\,R\,u$. Thus the relation is symmetric. The relation is reflexive because the loops guarantee that $u\,R\,u$ for each vertex u.

14. Since there are edges from Hawk to Crow, Owl, and Raccoon, the graph is telling us that the hawk competes with these three animals.

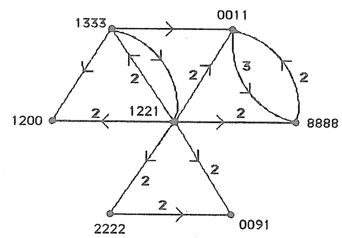
16. Each person is represented by a vertex, with an edge between two vertices if and only if the people are acquainted.



18. Fred influences Brian, since there is an edge from Fred to Brian. Yvonne and Deborah influence Fred, since there are edges from these vertices to Fred.

20. Team four beat the vertices to which there are edges from Team four, namely only Team three. The other teams—Team one, Team two, Team five, and Team six—all beat Team four, since there are edges from them to Team four.

22. This is a directed multigraph with one edge from a to b for each call made by a to b. Rather than draw the parallel edges with parallel lines, we have indicated what is intended by writing a numeral on the edge to indicate how many calls were made, if it was more than one.



24. This is similar to the use of directed graphs to model telephone calls.

a) We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.

b) As in part (a) we use a directed edge for each message sent during the week.

26. Vertices with thousands or millions of edges going out from them could be such lists.

28. We make the subway stations the vertices, with an edge from station u to station v if there is a train going from u to v without stopping. It is quite possible that some segments are one-way, so we should use directed edges. (If there are no one-way segments, then we could use undirected edges.) There would be no need for multiple edges, unless we had two kinds of edges, maybe with different colors, to represent local and express trains. In that case, there could be parallel edges of different colors between the same vertices, because both a local and an express train might travel the same segment. There would be no point in having loops, because no passenger would want to travel from a station back to the same station without stopping.

30. The model says that the statements for which there are edges to S_6 must be executed before S_6, namely the statements S_1, S_2, S_3, and S_4.

32. The vertices in the directed graph represent cities. Whenever there is a nonstop flight from city A to city B, we put a directed edge into our directed graph from vertex A to vertex B, and furthermore we label that edge with the flight time. Let us see how to incorporate this into the mathematical definition. Let us call such a thing a directed graph with weighted edges. It is defined to be a triple (V, E, W), where (V, E) is a directed graph (i.e., V is a set of vertices and E is a set of ordered pairs of elements of V) and W is a function from E to the set of nonnegative real numbers. Here we are simply thinking of $W(e)$ as the weight of edge e, which in this case is the flight time.

34. We can let the vertices represent people; an edge from u to v would indicate that u can send a message to v. We would need a directed multigraph in which the edges have labels, where the label on each edge indicates the form of communication (cell phone audio, text messaging, and so on).

SECTION 9.2 Graph Terminology and Special Types of Graphs

2. In this pseudograph there are 5 vertices and 13 edges. The degree of vertex a is 6, since in addition to the 4 nonloops incident to a, there is a loop contributing 2 to the degree. The degrees of the other vertices are $\deg(b) = 6$, $\deg(c) = 6$, $\deg(d) = 5$, and $\deg(e) = 3$. There are no pendant or isolated vertices in this pseudograph.

4. For the graph in Exercise 1, the sum is $2+4+1+0+2+3 = 12 = 2 \cdot 6$; there are 6 edges. For the pseudograph in Exercise 2, the sum is $6+6+6+5+3 = 26 = 2 \cdot 13$; there are 13 edges. For the pseudograph in Exercise 3, the sum is $3+2+4+0+6+0+4+2+3 = 24 = 2 \cdot 12$; there are 12 edges.

6. Model this problem by letting the vertices of a graph be the people at the party, with an edge between two people if they shake hands. Then the degree of each vertex is the number of people the person that vertex represents shakes hands with. By Theorem 1 the sum of the degrees is even (it is $2e$).

8. In this directed multigraph there are 4 vertices and 8 edges. The degrees are $\deg^-(a) = 2$, $\deg^+(a) = 2$, $\deg^-(b) = 3$, $\deg^+(b) = 4$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 1$.

10. For Exercise 7 the sum of the in-degrees is $3+1+2+1 = 7$, and the sum of the out-degrees is $1+2+1+3 = 7$; there are 7 edges. For Exercise 8 the sum of the in-degrees is $2+3+2+1 = 8$, and the sum of the out-degrees is $2+4+1+1 = 8$; there are 8 edges. For Exercise 9 the sum of the in-degrees is $6+1+2+4+0 = 13$, and the sum of the out-degrees is $1+5+5+2+0 = 13$; there are 13 edges.

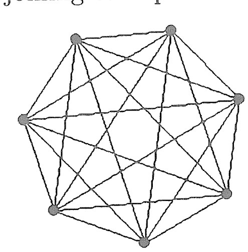
12. Since there is an edge from a person to each of his or her acquaintances, the degree of v is the number of people v knows. An isolated vertex would be a person who knew on one, and a pendant vertex would be a person who knew just one other person (it is doubtful that there are many, if any, isolated or pendant vertices). If the average degree is 1000, then the average person knows 1000 other people.

14. Since there is an edge from a person to each of the other actors that person has appeared with in a movie, the degree of v is the number of other actors that person has appeared with. An isolated vertex would be a person who has appeared only in movies in which he or she was the only actor, and a pendant vertex would be a person who has appeared with only one other actor in any movie (it is doubtful that there are many, if any, isolated or pendant vertices).

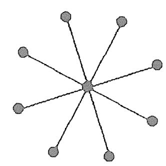
16. Since there is an edge from a page to each page that it links to, the outdegree of a vertex is the number of links on that page, and the in-degree of a vertex is the number of other pages that have a link to it.

18. This is essentially the same as Exercise 36 in Section 5.2, where the graph models the "know each other" relation on the people at the party. See the solution given for that exercise. The number of people a person knows is the degree of the corresponding vertex in the graph.

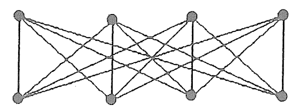
20. a) This graph has 7 vertices, with an edge joining each pair of distinct vertices.



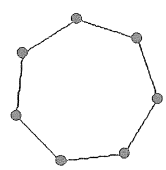
b) This graph is the complete bipartite graph on parts of size 1 and 8; we have put the part of size 1 in the middle.



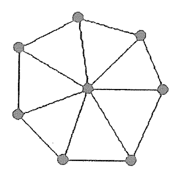
c) This is the complete bipartite graph with 4 vertices in each part.



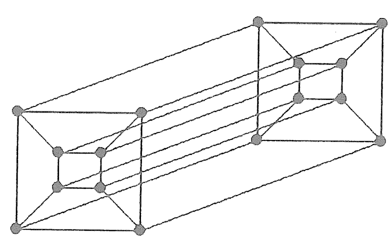
d) This is the 7-cycle.



e) The 7-wheel is the 7-cycle with an extra vertex joined to the other 7 vertices.



f) We take two copies of Q_3 and join corresponding vertices.



22. This graph is bipartite, with bipartition $\{a, c\}$ and $\{b, d, e\}$. In fact this is the complete bipartite graph $K_{2,3}$. If this graph were missing the edge between a and d, then it would still be bipartite on the same sets, but not a complete bipartite graph.

24. This is just like Exercise 22, but here we have the complete bipartite graph $K_{2,4}$. The vertices in the part of size 2 are c and f, and the vertices in the part of size 4 are a, b, d, and e.

26. a) By the definition given in the text, K_1 does not have enough vertices to be bipartite. Clearly K_2 is bipartite. There is a triangle in K_n for $n > 2$, so those complete graphs are not bipartite. (See Exercise 23.)
b) First we need $n \geq 3$ for C_n to be defined. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.
c) Every wheel contains triangles, so no W_n is bipartite.
d) Q_n is bipartite for all $n \geq 1$, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

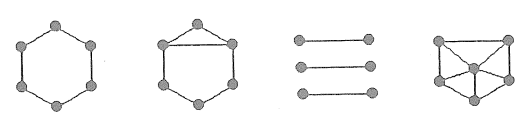
28. a) The partite sets are the set of women ({Anna, Barbara, Carol, Diane, Elizabeth}) and the set of men ({Jason, Kevin, Larry, Matt, Nick, Oscar}). We will use first letters for convenience. The given information tells us to have edges AJ, AL, AM, BK, BL, CJ, CN, CO, DJ, DL, DN, DO, EJ, and EM in our graph. We do not put an edge between a woman and a man she is not willing to marry.
b) By trial and error we easily find a matching (it's not unique), such as AL, BK, CJ, DN, and EM.

30. We just have to count the number of edges at each vertex, and then arrange these counts in nonincreasing order. For #21, we have $4,1,1,1,1$. For #22, we have $3,3,2,2,2$. For #23, we have $4,3,3,2,2,2$. For #24, we have $4,4,2,2,2,2$. For #25, we have $3,3,3,3,2,2$.

32. Assume that $m \geq n$. Then each of the n vertices in one part has degree m, and each of the m vertices in other part has degree n. Thus the degree sequence is $m, m, \ldots, m, n, n, \ldots, n$, where the sequence contains n copies of m and m copies of n. We put the m's first because we assumed that $m \geq n$. If $n \geq m$, then of course we would put the m copies of n first. If $m = n$, this would mean a total of $2n$ copies of n.

34. The 4-wheel (see Figure 5) with one edge along the rim deleted is such a graph. It has $(4+3+3+2+2)/2 = 7$ edges.

36. a) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.
b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.
c) A 6-cycle is such a graph. (See picture below.)
d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.
e) A 6-cycle with one of its diagonals added is such a graph. (See picture below.)
f) A graph consisting of three edges with no common vertices is such a graph. (See picture below.)
g) The 5-wheel is such a graph. (See picture below.)
h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.



(c) (e) (f) (g)

38. Since isolated vertices play no essential role, we can assume that $d_n > 0$. The sequence is graphic, so there is some simple graph G such that the degrees of the vertices are d_1, d_2, $\ldots$, d_n. Without loss of generality, we can label the vertices of our graph so that $d(v_i) = d_i$. Among all such graphs, choose G to be one in which v_1 is adjacent to as many of v_2, v_3, $\ldots$, v_{d_1+1} as possible. (The worst case might be that v_1 is not adjacent to any of these vertices.) If v_1 is adjacent to all of them, then we are done. We will show that if there is a vertex among v_2, v_3, $\ldots$, v_{d_1+1} that v_1 is not adjacent to, then we can find another graph with $d(v_i) = d_i$ and having v_1 adjacent to one more of the vertices v_2, v_3, $\ldots$, v_{d_1+1} than is true for G. This is a contradiction to the choice of G, and hence we will have shown that G satisfies the desired condition.

Under this assumption, then, let u be a vertex among v_2, v_3, $\ldots$, v_{d_1+1} that v_1 is not adjacent to, and let w be a vertex not among v_2, v_3, $\ldots$, v_{d_1+1} that v_1 is adjacent to; such a vertex w has to exist because $d(v_1) = d_1$. Because the degree sequence is listed in nonincreasing order, we have $d(u) \geq d(w)$. Consider all the vertices that are adjacent to u. It cannot be the case that w is adjacent to each of them, because then w would have a higher degree than u (because w is adjacent to v_1 as well, but u is not). Therefore there is some vertex x such that edge ux is present but edge xw is not present. Note also that edge v_1w is present but edge v_1u is not present. Now construct the graph G' to be the same as G except that edges ux and v_1w are removed and edges xw and v_1u are added. The degrees of all vertices are unchanged, but this graph has v_1 adjacent to more of the vertices among $v_2, v_3, \ldots, v_{d_1+1}$ than is the case in G. That gives the desired contradiction, and our proof is complete.

40. Given a sequence $d_1, d_2, \ldots, d_n$, if $n = 2$, then the sequence is graphic if and only if $d_1 = d_2 = 1$ (the graph consists of one edge)—this is one base case. Otherwise, if $n < d_1 + 1$, then the sequence is not graphic—this is the other base case. Otherwise (this is the recursive step), form a new sequence by deleting d_1, subtracting 1 from each of $d_2, d_3, \ldots, d_{d_1+1}$, deleting all 0's, and rearranging the terms into nonincreasing order. The original sequence is graphic if and only if the resulting sequence (with $n-1$ terms) is graphic.

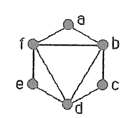
42. We list the subgraphs: the subgraph consisting of K_2 itself, the subgraph consisting of two vertices and no edges, and two subgraphs with 1 vertex each. Therefore the answer is 4.

44. We need to count this in an organized manner. First note that W_3 is the same as K_4, and it will be easier if we think of it as K_4. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 4 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(4,2) = 6$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $6 \cdot 2 = 12$ subgraphs with two vertices. If a subgraph is to have three vertices, then there are $C(4,3) = 4$ ways to choose the vertices, and then $2^3 = 8$ ways in each case to decide whether or not to include each of the edges joining pairs of them. This gives us $4 \cdot 8 = 32$ subgraphs with three vertices. Finally, there are the subgraphs containing all four vertices. Here there are $2^6 = 64$ ways to decide which edges to include. Thus our answer is $4 + 12 + 32 + 64 = 112$.

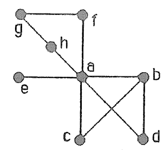
46. a) We want to show that $2e \geq vm$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot be less than the sum of m for each vertex, since each degree is no less than m.
b) We want to show that $2e \leq vM$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot exceed the sum of M for each vertex, since each degree is no greater than M.

48. Since the vertices in one part have degree m, and vertices in the other part have degree n, we conclude that $K_{m,n}$ is regular if and only if $m = n$.

50. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



52. The union is shown here. The only common vertex is a, so we have reoriented the drawing so that the pieces will not overlap.



54. The given information tells us that $G \cup \overline{G}$ has 28 edges. However, $G \cup \overline{G}$ is the complete graph on the number of vertices n that G has. Since this graph has $n(n-1)/2$ edges, we want to solve $n(n-1)/2 = 28$. Thus $n = 8$.

56. Following the ideas given in the solution to Exercise 57, we see that the degree sequence is obtained by subtracting each of these numbers from 4 (the number of vertices) and reversing the order. We obtain $2, 2, 1, 1, 0$.

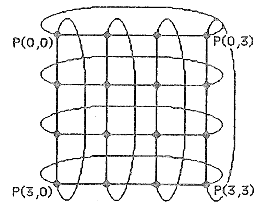
58. Suppose the parts are of sizes k and $v - k$. Then the maximum number of edges the graph may have is $k(v-k)$ (an edge between each pair of vertices in different parts). By algebra or calculus, we know that the function $f(k) = k(v-k)$ achieves its maximum when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

60. We start by coloring any vertex red. Then we color all the vertices adjacent to this vertex blue. Then we color all the vertices adjacent to blue vertices red, then color all the vertices adjacent to red vertices blue, and so on. If we ever are in the position of trying to color a vertex with the color opposite to the color it already has, then we stop and know that the graph is not bipartite. If the process terminates (successfully) before all the vertices have been colored, then we color some uncolored vertex red (it will necessarily not be adjacent to any vertices we have already colored) and begin the process again. Eventually we will have either colored all the vertices (producing the bipartition) or stopped and decided that the graph is not bipartite.

62. Obviously $(G^c)^c$ and G have the same vertex set, so we need only show that they have the same directed edges. But this is clear, since an edge (u, v) is in $(G^c)^c$ if and only if the edge (v, u) is in G^c if and only if the edge (u, v) is in G.

64. Let $|V_1| = n_1$ and $|V_2| = n_2$. Then the number of endpoints of edges in V_1 is $n \cdot n_1$, and the number of endpoints of edges in V_2 is $n \cdot n_2$. Since every edge must have one endpoint in each part, these two expressions must be equal, and it follows (because $n \neq 0$) that $n_1 = n_2$, as desired.

66. In addition to the connections shown in Figure 13, we need to make connections between $P(i, 3)$ and $P(i, 0)$ for each i, and between $P(3, j)$ and $P(0, j)$ for each j. The complete network is shown here. We can imagine this drawn on a torus.



SECTION 9.3 Representing Graphs and Graph Isomorphism

2. This is similar to Exercise 1. The list is as follows.

Vertex	Adjacent vertices
a	b, d
b	a, d, e
c	d, e
d	a, b, c
e	b, c

4. This is similar to Exercise 3. The list is as follows.

Initial vertex	Terminal vertices
a	b, d
b	a, c, d, e
c	b, c
d	a, e
e	c, e

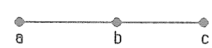
6. This is similar to Exercise 5. The vertices are assumed to be listed in alphabetical order.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

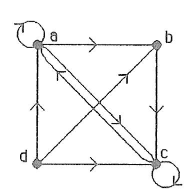
8. This is similar to Exercise 7.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

10. This graph has three vertices and is undirected, since the matrix is symmetric.



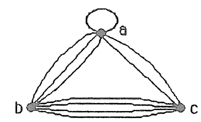
12. This graph is directed, since the matrix is not symmetric.



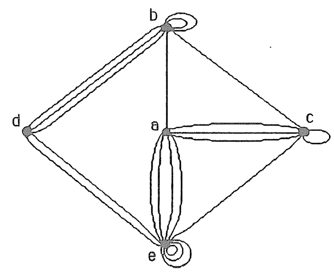
14. This is similar to Exercise 13.

$$\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

16. Because of the numbers larger than 1, we need multiple edges in this graph.



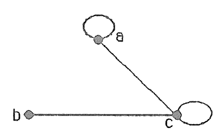
18. This is similar to Exercise 16.



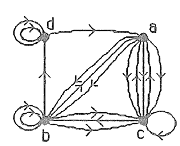
20. This is similar to Exercise 19.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

22. **a)** This matrix is symmetric, so we can take the graph to be undirected. No parallel edges are present, since no entries exceed 1.



24. This is the adjacency matrix of a directed multigraph, because the matrix is not symmetric and it contains entries greater than 1.



26. Each column represents an edge; the two 1's in the column are in the rows for the endpoints of the edge.

Exercise 1 $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ Exercise 2 $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

28. For an undirected graph, the sum of the entries in the i^{th} row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i. (See the solution to Exercise 29.)In a directed graph, the answer is dual to the answer for Exercise 29. The sum of the entries in the i^{th} row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

30. The sum of the entries in the i^{th} row of the incidence matrix is the number of edges incident to vertex i, since there is one column with a 1 in row i for each such edge.

32. a) This is just the matrix that has 0's on the main diagonal and 1's elsewhere, namely

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

b) We label the vertices so that the cycle goes $v_1, v_2, \ldots, v_n, v_1$. Then the matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

c) This matrix is the same as the answer in part (b), except that we add one row and column for the vertex in the middle of the wheel; in our matrix it is the last row and column:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}$$

d) Since the first m vertices are adjacent to none of the first m vertices but all of the last n, and vice versa, this matrix splits up into four pieces:

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

e) It is not convenient to show these matrices explicitly. Instead, we will give a recursive definition. Let $\mathbf{Q}_n$ be the adjacency matrix for the graph Q_n. Then

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbf{Q}_{n+1} = \begin{bmatrix} \mathbf{Q}_n & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{Q}_n \end{bmatrix},$$

where $\mathbf{I}_n$ is the identity matrix (since the corresponding vertices of the two n-cubes are joined by edges in the $(n+1)$-cube).

34. These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, and $f(u_5) = v_3$.

36. These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

38. These two graphs are isomorphic. Each consists of a K_4 with a fifth vertex adjacent to two of the vertices in the K_4. Many isomorphisms are possible. One is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_5$, and $f(u_5) = v_4$.

40. These graphs are not isomorphic—the degrees of the vertices are not the same (the graph on the right has a vertex of degree 4, which the graph on the left lacks).

42. These graphs are not isomorphic. In the first graph the vertices of degree 4 are adjacent. This is not true of the second graph.

44. The easiest way to show that these graphs are not isomorphic is to look at their complements. The complement of the graph on the left consists of two 4-cycles. The complement of the graph on the right is an 8-cycle. Since the complements are not isomorphic, the graphs are also not isomorphic.

46. This is immediate from the definition, since an edge is in $\overline{G}$ if and only if it is not in G, if and only if the corresponding edge is not in H, if and only if the corresponding edge is in $\overline{H}$.

48. An isolated vertex has no incident edges, so the row consists of all 0's.

50. The complementary graph consists of edges $\{a, c\}$, $\{c, d\}$, and $\{d, b\}$; it is clearly isomorphic to the original graph (send d to a, a to c, b to d, and c to b).

52. If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore the number of edges of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

54. **a)** There are just two graphs with 2 vertices—the one with no edges, and the one with one edge.

b) A graph with three vertices can contain 0, 1, 2, or 3 edges. There is only one graph for each number of edges, up to isomorphism. Therefore the answer is 4.

c) Here we look at graphs with 4 vertices. There is 1 graph with no edges, and 1 (up to isomorphism) with a single edge. If there are two edges, then these edges may or may not be adjacent, giving us 2 possibilities. If there are three edges, then the edges may form a triangle, a star, or a path, giving us 3 possibilities. Since graphs with four, five, or six edges are just complements of graphs with two, one, or no edges (respectively), the number of isomorphism classes must be the same as for these earlier cases. Thus our answer is $1 + 1 + 2 + 3 + 2 + 1 + 1 = 11$.

56. There are 9 such graphs. Let us first look at the graphs that have a cycle in them. There is only 1 with a 4-cycle. There are 2 with a triangle, since the fourth edge can either be incident to the triangle or not. If there are no cycles, then the edges may all be in one connected component (see Section 9.4), in which case there are 3 possibilities (a path of length four, a path of length three with an edge incident to one of the middle vertices on the path, and a star). Otherwise, there are two components, which are necessarily either two paths of length two, a path of length three plus a single edge, or a star with three edges plus a single edge (3 possibilities in this case as well).

58. **a)** These graphs are both K_3, so they are isomorphic.

b) These are both simple graphs with 4 vertices and 5 edges. Up to isomorphism there is only one such graph (its complement is a single edge), so the graphs have to be isomorphic.

60. We need only modify the definition of isomorphism of simple graphs slightly. The directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function $f : V_1 \to V_2$ such that for all pairs of vertices a and b in V_1, $(a,b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$.

62. These two graphs are not isomorphic. In the first there is no edge from the unique vertex of in-degree 0 (u_1) to the unique vertex of out-degree 0 (u_2), whereas in the second graph there is such an edge, namely v_3v_4.

64. We claim that the digraphs are isomorphic. To discover an isomorphism, we first note that vertices u_1, u_2, and u_3 in the first digraph are independent (i.e., have no edges joining them), as are u_4, u_5, and u_6. Therefore these two groups of vertices will have to correspond to similar groups in the second digraph, namely v_1, v_3, and v_5, and v_2, v_4, and v_6, in some order. Furthermore, u_3 is the only vertex among one of these groups of u's to be the only one in the group with out-degree 2, so it must correspond to v_6, the vertex with the similar property in the other digraph; and in the same manner, u_4 must correspond to v_5. Now it is an easy matter, by looking at where the edges lead, to see that the isomorphism (if there is one) must also pair up u_1 with v_2; u_2 with v_4; u_5 with v_1; and u_6 with v_3. Finally, we easily verify that this indeed gives an isomorphism—each directed edge in the first digraph is present precisely when the corresponding directed edge is present in the second digraph.

66. To show that the property that a graph is bipartite is an isomorphic invariant, we need to show that if G is bipartite and G is isomorphic to H, say via the function f, then H is bipartite. Let V_1 and V_2 be the partite sets for G. Then we claim that $f(V_1)$—the images under f of the vertices in V_1—and $f(V_2)$—the images under f of the vertices in V_2—form a bipartition for H. Indeed, since f must preserve the property of not being adjacent, since no two vertices in V_1 are adjacent, no two vertices in $f(V_1)$ are adjacent, and similarly for V_2.

68. a) There are 10 nonisomorphic directed graphs with 2 vertices. To see this, first consider graphs that have no edges from one vertex to the other. There are 3 such graphs, depending on whether they have no, one, or two loops. Similarly there are 3 in which there is an edge from each vertex to the other. Finally, there are 4 graphs that have exactly one edge between the vertices, because now the vertices are distinguished, and there can be or fail to be a loop at each vertex.

b) A detailed discussion of the number of directed graphs with 3 vertices would be rather long, so we will just give the answer, namely 104. There are some useful pictures relevant to this problem (and part (c) as well) in the appendix to *Graph Theory* by Frank Harary (Addison-Wesley, 1969).

c) The answer is 3069.

70. The answers depend on exactly how the storage is done, of course, but we will give naive answers that are at least correct as approximations.

a) We need one adjacency list for each vertex, and the list needs some sort of name or header; this requires v storage locations. In addition, each edge will appear twice, once in the list of each of its endpoints; this will require $2e$ storage locations. Therefore we need $v + 2e$ locations in all.

b) The adjacency matrix is a $v \times v$ matrix, so it requires v^2 bits of storage.

c) The incidence matrix is a $v \times e$ matrix, so it requires ve bits of storage.

SECTION 9.4 Connectivity

2. **a)** This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.

b) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

c) This is not a path, since there is no edge from d to b.

d) This is not a path, since there is no edge from b to d.

4. This graph is connected—it is easy to see that there is a path from every vertex to every other vertex.

6. The graph in Exercise 3 has three components: the piece that looks like a $\wedge$, the piece that looks like a $\vee$, and the isolated vertex. The graph in Exercise 4 is connected, with just one component. The graph in Exercise 5 has two components, each a triangle.

8. A connected component of a collaboration graph represent a maximal set of people with the property that for any two of them, we can find a string of joint works that takes us from one to the other. The word "maximal" here implies that nobody else can be added to this set of people without destroying this property.

10. An actor is in the same connected component as Kevin Bacon if there is a path from that person to Bacon. This means that the actor was in a movie with someone who was in a movie with someone who ... who was in a movie with Kevin Bacon. This includes Kevin Bacon, all actors who appeared in a movie with Kevin Bacon, all actors who appeared in movies with those people, and so on.

12. **a)** Notice that there is no path from f to a, so the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.

b) Notice that the sequence a, b, c, d, e, f, a provides a path from every vertex to every other vertex, so this graph is strongly connected.

c) The underlying undirected graph is clearly not connected (one component consists of the triangle), so this graph is neither strongly nor weakly connected.

14. **a)** The cycle $baeb$ guarantees that these three vertices are in one strongly connected component. Since there is no path from c to any other vertex, and there is no path from any other vertex to d, these two vertices are in strong components by themselves. Therefore the strongly connected components are $\{a, b, e\}$, $\{c\}$, and $\{d\}$.

b) The cycle $cdec$ guarantees that these three vertices are in one strongly connected component. The vertices a, b, and f are in strong components by themselves, since there are no paths both to and from each of these to every other vertex. Therefore the strongly connected components are $\{a\}$, $\{b\}$ $\{c, d, e\}$, and $\{f\}$.

c) The cycle $abcdfghia$ guarantees that these eight vertices are in one strongly connected component. Since there is no path from e to any other vertex, this vertex is in a strong component by itself. Therefore the strongly connected components are $\{a, b, c, d, f, g, h, i\}$ and $\{e\}$.

16. Let $a, b, c, \ldots, z$ be the directed path. Since z and a are in the same strongly connected component, there is a directed path from z to a. This path appended to the given path gives us a circuit. We can reach any vertex on the original path from any other vertex on that path by going around this circuit.

18. The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

20. We notice that there are two vertices in each graph that are not in cycles of size 4. So let us try to construct an isomorphism that matches them, say $u_1 \leftrightarrow v_2$ and $u_8 \leftrightarrow v_6$. Now u_1 is adjacent to u_2 and u_3, and v_2 is adjacent to v_1 and v_3, so we try $u_2 \leftrightarrow v_1$ and $u_3 \leftrightarrow v_3$. Then since u_4 is the other vertex adjacent to u_3 and v_4 is the other vertex adjacent to v_3 (and we already matched u_3 and v_3), we must have $u_4 \leftrightarrow v_4$. Proceeding along similar lines, we then complete the bijection with $u_5 \leftrightarrow v_8$, $u_6 \leftrightarrow v_7$, and $u_7 \leftrightarrow v_5$. Having thus been led to the only possible isomorphism, we check that the 12 edges of G exactly correspond to the 12 edges of H, and we have proved that the two graphs are isomorphic.

22. **a)** Adjacent vertices are in different parts, so every path between them must have odd length. Therefore there are no paths of length 2.

b) A path of length 3 is specified by choosing a vertex in one part for the second vertex in the path and a vertex in the other part for the third vertex in the path (the first and fourth vertices are the given adjacent vertices). Therefore there are $3 \cdot 3 = 9$ paths.

c) As in part **(a)**, the answer is 0.

d) This is similar to part **(b)**; therefore the answer is $3^4 = 81$.

24. Probably the best way to do this is to write down the adjacency matrix for this graph and then compute its powers. The matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

a) To find the number of paths of length 2, we need to look at $\mathbf{A}^2$, which is

$$\begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 3 & 2 & 2 \\ 2 & 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 2 & 3 & 1 & 4 & 1 \\ 2 & 2 & 1 & 2 & 1 & 3 \end{bmatrix}.$$

Since the $(3,4)^{\text{th}}$ entry is 0, so there are no paths of length 2.

b) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^3$ turns out to be 8, so there are 8 paths of length 3.

c) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^4$ turns out to be 10, so there are 10 paths of length 4.

d) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^5$ turns out to be 73, so there are 73 paths of length 5.

e) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^6$ turns out to be 160, so there are 160 paths of length 6.

f) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^7$ turns out to be 739, so there are 739 paths of length 7.

26. We show this by induction on n. For $n = 1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degrees is less than $2n$, which is less than $2(n+1)$. Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

28. Let v be a vertex of odd degree, and let H be the component of G containing v. Then H is a graph itself, so it has an even number of vertices of odd degree. In particular, there is another vertex w in H with odd degree. By definition of connectivity, there is a path from v to w.

30. Vertices c and d are the cut vertices. The removal of either one creates a graph with two components. The removal of any other vertex does not disconnect the graph.

32. The graph in Exercise 29 has no cut edges; any edge can be removed, and the result is still connected. For the graph in Exercise 30, $\{c, d\}$ is the only cut edge. There are several cut edges for the graph in Exercise 31: $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{c, e\}$, $\{e, i\}$, and $\{h, i\}$.

34. First we show that if c is a cut vertex, then there exist vertices u and v such that every path between them passes through c. Since the removal of c increases the number of components, there must be two vertices in G that are in different components after the removal of c. Then every path between these two vertices has to pass through c. Conversely, if u and v are as specified, then they must be in different components of the graph with c removed. Therefore the removal of c resulted in at least two components, so c is a cut vertex.

36. First suppose that $e = \{u, v\}$ is a cut edge. Every circuit containing e must contain a path from u to v in addition to just the edge e. Since there are no such paths if e is removed from the graph, every such path must contain e. Thus e appears twice in the circuit, so the circuit is not simple. Conversely, suppose that e is not a cut edge. Then in the graph with e deleted u and v are still in the same component. Therefore there is a simple path P from u to v in this deleted graph. The circuit consisting of P followed by e is a simple circuit containing e.

38. (The answers given here are not unique.) In the directed graph in Exercise 7, there is a path from b to each of the other three vertices, so $\{b\}$ is a vertex basis (and a smallest one). For the directed graph in Exercise 8, there is a path from b to each of a and c; on the other hand, d must clearly be in every vertex basis. Thus $\{b, d\}$ is a smallest vertex basis. Every vertex basis for the directed graph in Exercise 9 must contain vertex e, since it has no incoming edges. On the other hand, from any other vertex we can reach all the other vertices, so e together with any one of the other four vertices will form a vertex basis.

40. By definition of graph, both G_1 and G_2 are nonempty. If they have no common vertex, then there clearly can be no paths from $v_1 \in G_1$ to $v_2 \in G_2$. In that case G would not be connected, contradicting the hypothesis.

42. First we obtain the inequality given in the hint. We claim that the maximum value of $\sum n_i^2$, subject to the constraint that $\sum n_i = n$, is obtained when one of the n_i's is as large as possible, namely $n - k + 1$, and the remaining n_i's (there are $k - 1$ of them) are all equal to 1. To justify this claim, suppose instead that two of the n_i's were a and b, with $a \geq b \geq 2$. If we replace a by $a + 1$ and b by $b - 1$, then the constraint is still satisfied, and the sum of the squares has changed by $(a+1)^2 + (b-1)^2 - a^2 - b^2 = 2(a-b) + 2 \geq 2$. Therefore the maximum cannot be attained unless the n_i's are as we claimed. Since there are only a finite number of possibilities for the distribution of the n_i's, the arrangement we give must in fact yield the maximum. Therefore $\sum n_i^2 \leq (n - k + 1)^2 + (k - 1) \cdot 1^2 = n^2 - (k - 1)(2n - k)$, as desired.

Now by Exercise 41, the number of edges of the given graph does not exceed $\sum C(n_i, 2) = \sum (n_i^2 + n_i)/2 = ((\sum n_i^2) + n)/2$. Applying the inequality obtained above, we see that this does not exceed $(n^2 - (k-1)(2n - k) + n)/2$, which after a little algebra is seen to equal $(n - k)(n - k + 1)/2$. The upshot of all this is that the most edges are obtained if there is one component as large as possible, with all the other components consisting of isolated vertices.

44. Under these conditions, the matrix has a block structure, with all the 1's confined to small squares (of various sizes) along the main diagonal. The reason for this is that there are no edges between different components. See the picture for a schematic view. The only 1's occur inside the small submatrices (but not all the entries in these squares are 1's, of course).

$$\begin{bmatrix} [\;] & & & & 0 \\ & [\,] & & & \\ & & [\;] & & \\ & & & [\,] \ddots & \\ 0 & & & & [\;] \end{bmatrix}$$

46. The length of a shortest path is the smallest l such that there is at least one path of length l from v to w. Therefore we can find the length by computing successively $\mathbf{A}^1$, $\mathbf{A}^2$, $\mathbf{A}^3$, ..., until we find the first l such that the $(i,j)^{\text{th}}$ entry of $\mathbf{A}^l$ is not 0, where v is the i^{th} vertex and w is the j^{th}.

48. First we write down the adjacency matrix for this graph, namely

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then we compute $\mathbf{A}^2$ and $\mathbf{A}^3$, and look at the $(1,3)^{\text{th}}$ entry of each. We find that these entries are 0 and 1, respectively. By the reasoning given in Exercise 47, we conclude that a shortest path has length 3.

50. Suppose that f is an isomorphism from graph G to graph H. If G has a simple circuit of length k, say $u_1, u_2, \ldots, u_k, u_1$, then we claim that $f(u_1), f(u_2), \ldots, f(u_k), f(u_1)$ is a simple circuit in H. Certainly this is a circuit, since each edge u_iu_{i+1} (and u_ku_1) in G corresponds to an edge $f(u_i)f(u_{i+1})$ (and $f(u_k)f(u_1)$) in H. Furthermore, since no edge was repeated in this circuit in G, no edge will be repeated when we use f to move over to H.

52. The adjacency matrix of G is as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute $\mathbf{A}^2$ and $\mathbf{A}^3$, obtaining

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^3 = \begin{bmatrix} 2 & 3 & 5 & 2 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 & 1 & 2 & 1 \\ 5 & 5 & 4 & 6 & 1 & 6 & 1 \\ 2 & 2 & 6 & 2 & 3 & 5 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 2 & 6 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \end{bmatrix}.$$

Already every off-diagonal entry in $\mathbf{A}^3$ is nonzero, so we know that there is a path of length 3 between every pair of distinct vertices in this graph. Therefore the graph G is connected.

On the other hand, the adjacency matrix of H is as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We compute $\mathbf{A}^2$ through $\mathbf{A}^5$, obtaining the following matrices:

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}^3 = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 & 2 \end{bmatrix}$$

$$\mathbf{A}^4 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 5 & 5 \\ 0 & 0 & 0 & 5 & 6 & 5 \\ 0 & 0 & 0 & 5 & 5 & 6 \end{bmatrix} \qquad \mathbf{A}^5 = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 11 & 11 \\ 0 & 0 & 0 & 11 & 10 & 11 \\ 0 & 0 & 0 & 11 & 11 & 10 \end{bmatrix}$$

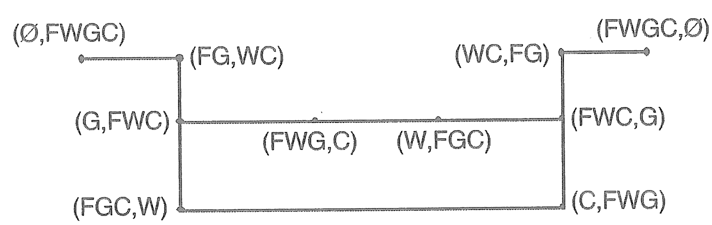
If we compute the sum $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4 + \mathbf{A}^5$ we obtain

$$\begin{bmatrix} 6 & 7 & 7 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 21 & 21 \\ 0 & 0 & 0 & 21 & 20 & 21 \\ 0 & 0 & 0 & 21 & 21 & 20 \end{bmatrix}.$$

There is a 0 in the $(1,4)$ position, telling us that there is no path of length at most 5 from vertex a to vertex d. Since the graph only has six vertices, this tells us that there is no path at all from a to d. Thus the fact that there was a 0 as an off-diagonal entry in the sum told us that the graph was not connected.

54. a) To proceed systematically, we list the states in order of decreasing population on the left shore. The allowable states are then $(FWGC,\emptyset)$, (FWG,C), (FWC,G), (FGC,W), (FG,WC), (WC,FG) (C,FWG), (G,FWC), (W,FGC), and $(\emptyset,FWGC)$. Notice that, for example, (GC,FW) and (WGC,F) are not allowed by the rules.

b) The graph is as shown here. Notice that the boat can carry only the farmer and one other object, so the transitions are rather restricted.



c) The path in the graph corresponds to the moves in the solution.

d) There are two simple paths from $(FWGC,\emptyset)$ to $(\emptyset,FWGC)$ that can be easily seen in the graph. One is $(FWGC,\emptyset)$, (WC,FG), (FWC,G), (W,FGC), (FWG,C), (G,FWC), (FG,WC), $(\emptyset,FWGC)$. The other is $(FWGC,\emptyset)$, (WC,FG), (FWC,G), (C,FWG), (FGC,W), (G,FWC), (FG,WC), $(\emptyset,FWGC)$.

e) Both solutions cost \$4.

56. If we use the ordered pair (a,b) to indicate that the three-gallon jug has a gallons in it and the five-gallon jug has b gallons in it, then we start with $(0,0)$ and can do the following things: fill a jug that is empty or partially empty (so that, for example, we can go from $(0,3)$ to $(3,3)$); empty a jug; or transfer some or all of the contents of a jug to the other jug , as long as we either completely empty the donor jug or completely fill the receiving jug. A simple solution to the puzzle uses this directed path: $(0,0) \to (3,0) \to (0,3) \to (3,3) \to (1,5)$.

SECTION 9.6 Shortest-Path Problems

2. In the solution to Exercise 5 we find a shortest path. Its length is 7.

4. In the solution to Exercise 5 we find a shortest path. Its length is 16.

6. The solution to this problem is given in the solution to Exercise 7, where the paths themselves are found.

8. In theory, we can use Dijkstra's algorithm. In practice with graphs of this size and shape, we can tell by observation what the conceivable answers will be and find the one that produces the minimum total length by inspection.

a) The direct path is the shortest.
b) The path via Chicago only is the shortest.
c) The path via Atlanta and Chicago is the shortest.
d) The path via Atlanta, Chicago and Denver is the shortest.

10. The comments for Exercise 8 apply.

a) The direct flight is the cheapest.
b) The path via New York is the cheapest.
c) The path via New York and Chicago is the cheapest.
d) The path via New York is the cheapest.

12. The comments for Exercise 8 apply.

a) The path through Chicago is the fastest.
b) The path via Chicago is the fastest.
c) The path via Denver (or the path via Los Angeles) is the fastest.
d) The path via Dallas (or the path via Chicago) is the fastest.

14. Here we simply assign the weight of 1 to each edge.

16. We need to keep track of the vertex from which a shortest path known so far comes, as well as the length of that path. Thus we add an array P to the algorithm, where $P(v)$ is the previous vertex in the best known path to v. We modify Algorithm 1 so that when L is updated by the statement $L(v) := L(u) + w(u, v)$, we also set $P(v) := u$. Once the **while** loop has terminated, we can obtain a shortest path from a to z *in reverse* by starting with z and following the pointers in P. Thus the path in reverse is z, $P(z)$, $P(P(z))$, $\ldots$, $P(P(\cdots P(z)\cdots)) = a$.

18. The shortest path need not be unique. For example, we could have a graph with vertices a, b, c, and d, with edges $\{a, b\}$ of weight 3, $\{b, c\}$ of weight 7, $\{a, d\}$ of weight 4, and $\{d, c\}$ of weight 6. There are two shortest paths from a to c.

20. We give an ad hoc analysis. Recall that a simple path cannot use any edge more than once. Furthermore, since the path must use an odd number of edges incident to a and an odd number of edges incident to z, the path must omit at least two edges, one at each end. The best we could hope for, then, in trying for a path of maximum length, is that the path leaves out the shortest such edges—$\{a, c\}$ and $\{e, z\}$. If the path leaves out these two edges, then it must also leave out one more edge incident to c, since the path must use an even number of the three remaining edges incident to c. The best we could hope for is that the path omits the two aforementioned edges and edge $\{b, c\}$. Since $2 + 1 < 4$, this is better than the other possibility, namely omitting edge $\{a, b\}$ instead of edge $\{a, c\}$. Finally, we find a simple path omitting only these three edges, namely a, b, d, c, e, d, z, with length 35, and thus we conclude that it is a longest simple path from a to z.

A similar argument shows that the longest simple path from c to z is c, a, b, d, c, e, d, z

22. It follows by induction on i that after the i^{th} pass through the triply nested **for** loop in the pseudocode, $d(v_j, v_k)$ gives, for each j and k, the shortest distance between v_j and v_k using only intermediate vertices v_m for $m \le i$. Therefore after the final path, we have obtained the shortest distance.

24. Consider the graph with vertices a, b, and z, where the weight of $\{a, z\}$ is 2, the weight of $\{a, b\}$ is 3, and the weight of $\{b, z\}$ is -2. Then Dijkstra's algorithm will decide that $L(z) = 2$ and stop, whereas the path a, b, z is shorter (has length 1).

26. The following table shows the twelve different Hamilton circuits and their weights:

Circuit	Weight
a-b-c-d-e-a	$3 + 10 + 6 + 1 + 7 = 27$
a-b-c-e-d-a	$3 + 10 + 5 + 1 + 4 = 23$
a-b-d-c-e-a	$3 + 9 + 6 + 5 + 7 = 30$
a-b-d-e-c-a	$3 + 9 + 1 + 5 + 8 = 26$
a-b-e-c-d-a	$3 + 2 + 5 + 6 + 4 = 20$
a-b-e-d-c-a	$3 + 2 + 1 + 6 + 8 = 20$
a-c-b-d-e-a	$8 + 10 + 9 + 1 + 7 = 35$
a-c-b-e-d-a	$8 + 10 + 2 + 1 + 4 = 25$
a-c-d-b-e-a	$8 + 6 + 9 + 2 + 7 = 32$
a-c-e-b-d-a	$8 + 5 + 2 + 9 + 4 = 28$
a-d-b-c-e-a	$4 + 9 + 10 + 5 + 7 = 35$
a-d-c-b-e-a	$4 + 6 + 10 + 2 + 7 = 29$

Thus we see that the circuits a-b-e-c-d-a and a-b-e-d-c-a (or the same circuits starting at some other point but traversing the vertices in the same or exactly opposite order) are the ones with minimum total weight.

28. The following table shows the twelve different Hamilton circuits and their weights, where we abbreviate the cities with the beginning letter of their name, except that New Orleans is O:

Circuit	Weight
S-B-N-O-P-S	$409 + 109 + 229 + 309 + 119 = 1175$
S-B-N-P-O-S	$409 + 109 + 319 + 309 + 429 = 1575$
S-B-O-N-P-S	$409 + 239 + 229 + 319 + 119 = 1315$
S-B-O-P-N-S	$409 + 239 + 309 + 319 + 389 = 1665$
S-B-P-N-O-S	$409 + 379 + 319 + 229 + 429 = 1765$
S-B-P-O-N-S	$409 + 379 + 309 + 229 + 389 = 1715$
S-N-B-O-P-S	$389 + 109 + 239 + 309 + 119 = 1165$
S-N-B-P-O-S	$389 + 109 + 379 + 309 + 429 = 1615$
S-N-O-B-P-S	$389 + 229 + 239 + 379 + 119 = 1355$
S-N-P-B-O-S	$389 + 319 + 379 + 239 + 429 = 1755$
S-O-B-N-P-S	$429 + 239 + 109 + 319 + 119 = 1215$
S-O-N-B-P-S	$429 + 229 + 109 + 379 + 119 = 1265$

As a check of our arithmetic, we can compute the total weight (price) of all the trips (it comes to 17580) and check that it is equal to 6 times the sum of the weights (which here is 2930), since each edge appears in six paths (and sure enough, $17580 = 6 \cdot 2930$). We see that the circuit S-N-B-O-P-S (or the same circuit starting at some other point but traversing the vertices in the same or exactly opposite order) is the one with minimum total weight, 1165.