First-Order Logic

Chapter 8

Problem of Propositional Logic

☹ Propositional logic has very limited expressive power

– E.g., cannot say "pits cause breezes in adjacent squares“ except by writing one sentence for each square.

– We want to be able to say this in one single sentence: “for all squares and pits, pits cause breezes in adjacent squares.

– First order logic will provide this flexibility.

First-order logic

- Propositional logic assumes the world contains facts that are true or false.

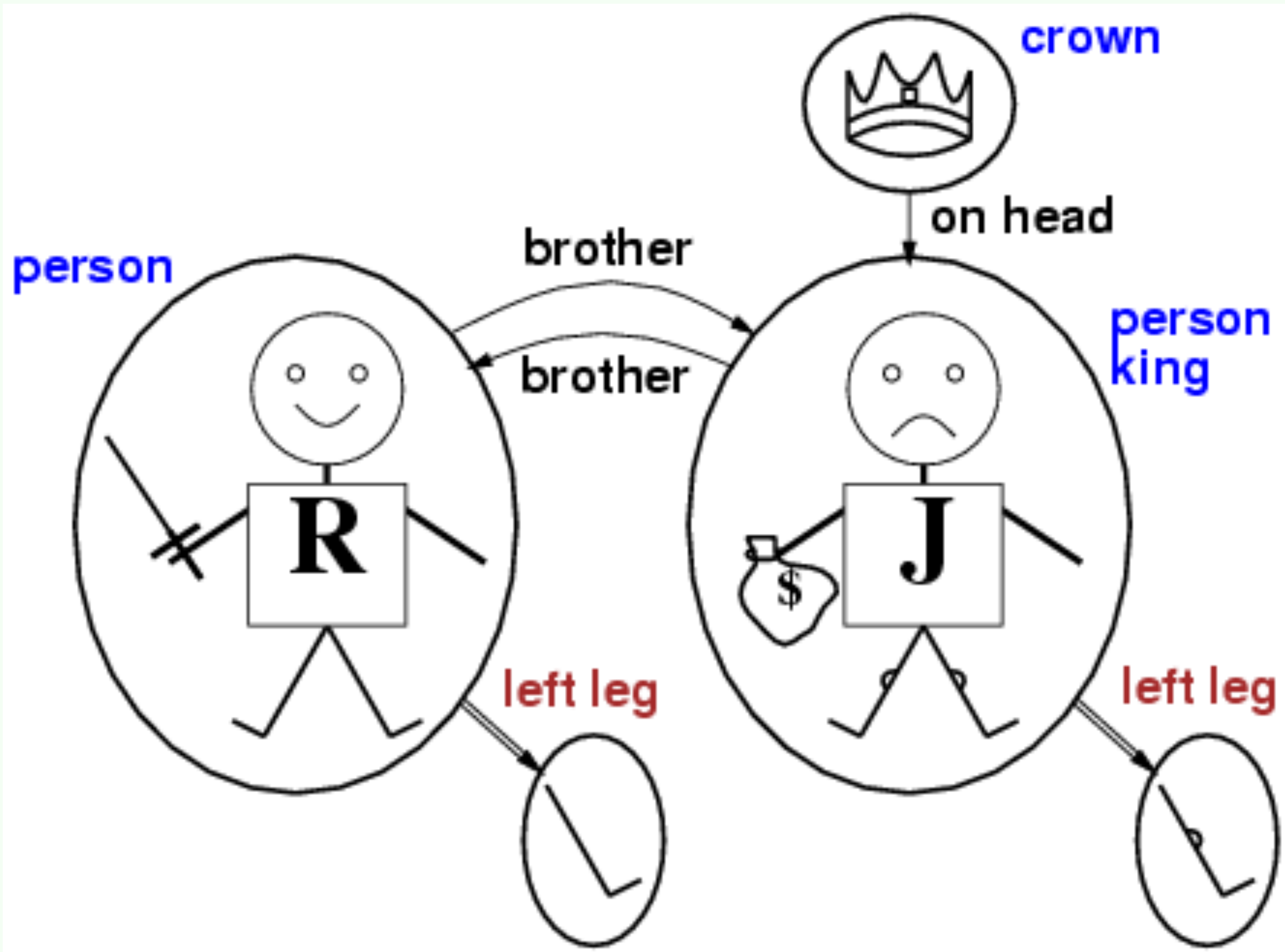
- First-order logic

  assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations between objects: red, round, prime, brother of, bigger than, part of, comes between, …

Relations

- Some relations are properties: they state some fact about a single object: Round(ball), Prime(7).
- n-ary relations state facts about two or more objects: Married(John,Mary), Largerthan(3,2).
- Some relations are functions: their value is another object: Plus(2,3), Father(Dan).

Models for FOL: Example



Atomic Sentences

- Sentences in logic state facts that are true or false.

- Properties and n-ary relations do just that:

  LargerThan(2,3) (means 2>3) is false.

  Brother(Mary,Pete) is false.

- Note: Functions do not state facts and form no sentence: Brother(Pete) refers to the object John (his brother) and is neither true nor false.

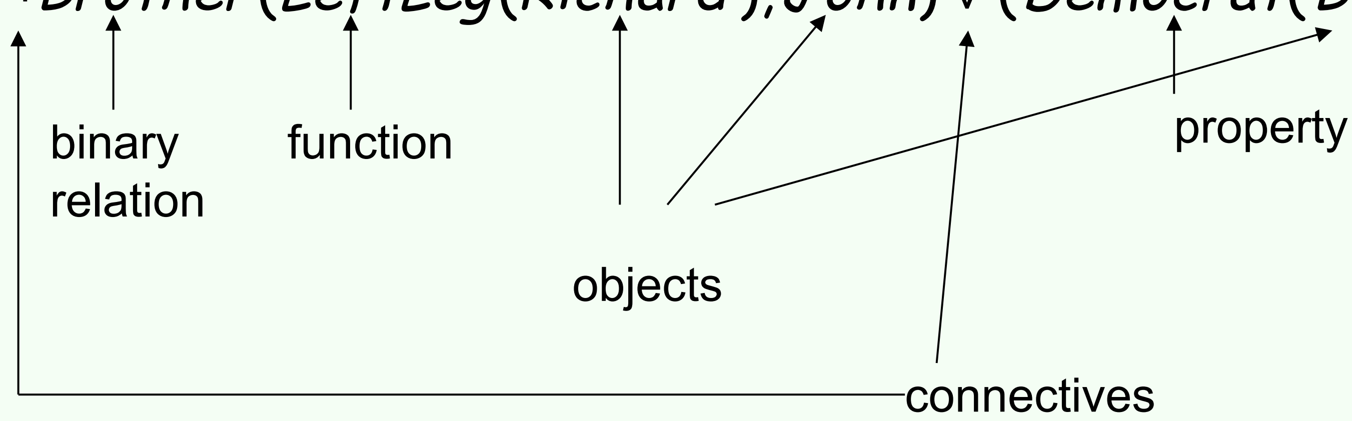
- Brother(Pete,Brother(Pete)) is True.

Binary relation   Function

Complex Sentences

- We make complex sentences with connectives (just like in proposition logic).

$$\neg Brother(LeftLeg(Richard), John) \vee (Democrat(Bush))$$



Quantification

- Round(ball) is true or false because we give it a single argument (ball).

- We can be much more flexible if we allow variables which can take on values in a domain. e.g. reals x, all persons P, etc.

- To construct logical sentences we need a quantifier to make it true or false.

Quantifier

- Is the following true or false? $x > 5,\ x \in R$

- To make it true or false we use $\forall$ *and* $\exists$

$$\forall x[(x > 2) \Rightarrow (x > 3)] \quad x \in R \quad (false)$$

$$\exists x[(x^2 = -1)] \quad x \in R \quad (false)$$

For all real x, x>2 implies x>3.

There exists some real x which square is minus 1.

Nested Quantifiers

- Combinations of universal and existential quantification are possible:

$$\forall x \forall y\, Father(x, y) \equiv \forall y \forall x\, Father(x, y)$$

$$\exists x \exists y\, Father(x, y) \equiv \exists y \exists x\, Father(x, y)$$

$$\forall x \exists y\, Father(x, y) \neq \exists y \forall x\, Father(x, y)$$

$$\exists x \forall y\, Father(x, y) \neq \forall y \exists x\, Father(x, y)$$

$$x, y \in \{All\ \ people\}$$

Binary relation:
"x is a father of y".

Quiz :which is which: Everyone is the father of someone.
Everyone has everyone as a father
There is a person who has everyone as a father.
There is a person who has a father
There is a person who is the father of everyone.
Everyone has a father.

De Morgan's Law for Quantifiers

De Morgan's Rule

$$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$$

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Generalized De Morgan's Rule

$$\forall x\, P \equiv \neg \exists x\, (\neg P)$$

$$\exists x\, P \equiv \neg \forall x\, (\neg P)$$

$$\neg \forall x\, P \equiv \exists x\, (\neg P)$$

$$\neg \exists x\, P \equiv \forall x\, (\neg P)$$

Rule is simple: if you bring a negation inside a disjunction or a conjunction, always switch between them (or →and, and → or).

- Equality symbol: Father(John)=Henry.
  This relates two objects.

Common mistakes to avoid

- $\Rightarrow$ is the main connective with $\forall$
- $\wedge$ is the main connective with $\exists$

$$\forall x, King(x) \Rightarrow Person(x) \quad x = \{Pete, Mary, tablespoon\}$$

$$\forall x, King(x) \wedge Person(x)$$

$$\exists x, King(x) \Rightarrow Person(x)$$

$$\exists x, King(x) \wedge Person(x)$$

All of these must be true!
King(Pete) AND Person(Pete)
King(Mary) AND Person(Mary)
King(Tablespoon) AND Person(Tablespoon)

too strong False!

One of these should be true!
if King(Pete) then Person(Pete)
if King(Mary) then Person(Mary)
If King(Tablespoon) then Person(Tablespoon) True!

too weak

Using FOL

- We want to TELL things to the KB, e.g. TELL(KB, $\forall x, King(x) \Rightarrow Person(x)$)

- We also want to ASK things to the KB, ASK(KB, $\exists x, Person(x)$)

- The KB should return the list of x's for which Person(x) is true: {x/John,x/Richard,...}

Examples

The kinship domain:

- Brothers are siblings

  $\forall$x,y *Brother(x,y)* => *Sibling(x,y)*

- One's mother is one's female parent

  $\forall$m,c *Mother(c)* = m $\Leftrightarrow$ *(Female(m)* $\wedge$ *Parent(m,c))*

- “Sibling” is symmetric

  $\forall$x,y *Sibling(x,y)* $\Leftrightarrow$ *Sibling(y,x)*

Some may be considered axioms, others as theorems which can be derived from the axioms.

Translating English to FOL

- Every gardener likes the sun.
  `(Ax) gardener(x) => likes(x,Sun)`

- You can fool some of the people all of the time.
  `(Ex)(At) (person(x) ^ time(t)) => can-fool(x,t)`

- You can fool all of the people some of the time.
  `(Ax)(Et) (person(x) ^ time(t) => can-fool(x,t)`

- All purple mushrooms are poisonous.
  `(Ax) (mushroom(x) ^ purple(x)) => poisonous(x)`

- No purple mushroom is poisonous.
  `~(Ex) purple(x) ^ mushroom(x) ^ poisonous(x)`
  or, equivalently,
  `(Ax) (mushroom(x) ^ purple(x)) => ~poisonous(x)`

- There are exactly two purple mushrooms.
  `(Ex)(Ey) mushroom(x) ^ purple(x) ^ mushroom(y) ^ purple(y) ^ ~(x=y) ^ (Az) (mushroom(z) ^ purple(z)) => ((x=z) v (y=z))`

1.(12 pts) **First Order Logic**

Consider the following sentence in FOL:

$$\forall x \text{ Married}(\text{Father}(\text{x}), \text{Mother}(\text{x})) \Rightarrow$$
$$\exists y \text{ Certificate}(y) \wedge \text{Names}(y, \text{Father}(x), \text{Mother}(x))$$

In English: *For every person who has a father and a mother that are married, there exists a paper which is a wedding certificate and which contains the names of both the father and the mother of this person.*

a.(4 pts) Identify the *functions, properties, binary relations, relations with arity 3, quantifiers and connectives* in this sentence.

b.(4 pts) Consider the sentence: $\exists x\ [p(x) \Rightarrow q(x)]$. Assume we know that there is no value for x for which $q(x)$ is true. Is it still possible that the above sentence is true?

c.(4 pts) Provide the truth-table for $(P \vee Q) \Rightarrow (P \wedge Q)$ where P and Q are two propositions.

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a) answer: Function: Father, Mother, Property: Certificate, Binary Relation.: Married, Tertiary Relation: Names, Quantifiers: $\forall, \exists$, Connectives: $\rightarrow, \wedge$.

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b) answer: P:T,T,F,F, Q:T,F,T,F, $(P \vee Q) \Rightarrow (P \wedge Q)$:T,F,F,T.

3.(10 pts) **First Order Logic**

Consider the following relations: $F(x)$ is true when x is female, $M(x)$ is true when x is male, $D(x)$ is true when x lives in Disneyland and $L(x, y)$ is true when x likes y. Translate the following sentences into first order logic:

a.(5 pts) There is at least one male and female, both living in Disneyland, that like each other.

b.(5 pts) All males and females living in Disneyland like each other.

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b) answer: $\forall x, y\ M(x) \wedge F(y) \wedge D(x, y) \Longrightarrow L(x, y) \wedge L(y, x)$.